Public schooling, college subsidies and growth∗

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Abstract
How does the mix of public education expenditures across primary and secondary (k-12) education and post-secondary (college) education influence economic growth? To answer this, I build an overlapping generations endogenous growth model. Human capital is accumulated through compulsory k-12 education and through optional college education. Government uses tax revenue to provide quality in k-12 schooling and to subsidize college tuition. When total expenditures are small, all funds should provide k-12 quality. When expenditures are above a critical value, a positive share should subsidize tuition in order to maximize growth. The share should be larger when total expenditures are larger and when human capital accumulated through k-12 and college education are more complementary in production. Also, increased education spending is more likely to increase growth when a larger share subsidizes tuition.

Keywords: Endogenous Growth; Education Subsidies; Human Capital.

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1 Introduction.

Government is the principal source of funding for primary and secondary (k-12) education in the United States. More than 90% of expenditures are financed by federal, state, and local governments.\(^1\) Government’s role in financing post-secondary (college) education is substantial but smaller. Private sources account for nearly half of total funding.\(^2\) Public k-12 expenditures directly affect a large majority of the school-aged population. For those under age 16, k-12 enrollment is near 100% and more than 88% of k-12 students enroll in public schools.\(^3\) The impressive enrollment figures reflect mandatory attendance laws in each of the 50 states.\(^4\) In contrast, no state requires schooling beyond the high school level and college enrollment is much lower.\(^5\) To a first approximation the U.S. educational system is one of mandatory, publicly funded k-12 education and discretionary college education with partial public funding.

The choice of public education expenditures, then, has two dimensions. Policy-makers must decide upon both the level of public expenditures and its allocation across k-12 and college education. Previous investigations of the growth effects of public education funding have largely focused on the first decision. This literature has taken two broad approaches to modeling spending. In several papers, expenditures convert to quality of education which is a direct input in the production of human capital. Examples include Glomm and Ravikumar (1992, 1997, 1998),

\(^{\text{1}}\)For ease of exposition, I use ‘k-12’ (kindergarten through twelfth grade) and ‘college’ to describe mandatory, general education and optional specialized education. The latter can include any post-secondary education.

\(^{\text{2}}\)1998 figures. Public k-12 expenditures accounted for 3.4% of GDP. Public college expenditures were less than 40% of this. See Education at a Glance [2001], tables B3.2 and B4.1.

\(^{\text{3}}\)The enrollment rate for older high school-aged students is also high and many who drop out eventually finish. In 1999, more than 86% of 18-29 year-olds had completed high school. Digest of Education Statistics [2000], tables 3 and 105.

\(^{\text{4}}\)30 states require attendance to age 16. The remainder require attendance to age 17 or 18. Digest [1999], table 155. Angrist and Krueger (1991) find that about 25% of potential dropouts remain in school as a result of such laws.

\(^{\text{5}}\)Of those who complete high school, only about 62% enroll in a post-secondary education institution. Many enrollees never complete a degree program. Of those first-time post secondary students enrolled in 4-year institutions in the 89-90 school year, only 53.3% had finished a bachelor’s degree by the Spring of 1994. Nearly a quarter had no degree and were no longer enrolled. Many had finished with certificates or associates degrees. Digest [2000], table 311.
Eckstein and Zilcha (1994) Kaganovich and Zilcha (1999) and Cassou and Lansing (2001). In other work, public expenditures subsidize private spending. This feature is found, for example, in Zhang (1996), Milesi-Ferretti and Roubini (1998), Hendricks (1999) and Brauninger and Vidal (2000). In the U.S., direct government provision of education resources resembles k-12 education most closely while subsidies are most meaningful in the college education decision. Thus important aspects of each type of public education expenditures are captured in these models. Several papers contrast publicly provided education with subsidies to private education. However, none simultaneously allow public inputs to k-12 education and subsidies of private college education expenditures. Thus the models are silent on how the mix of expenditures across these uses may influence growth.

There are several reasons to suspect the mix may be important. First, since k-12 education is mandatory, expenditures do not directly promote enrollment. If they influence the level of human capital, it is through education quality. In contrast, funding college education in part lowers the private cost to encourage a greater quantity of education. Secondly, human capital accumulated through k-12 and college education may be imperfect substitutes in production. Thus the effects of increasing the supply of each may not be symmetric.

This paper addresses the growth effects of each decision. In particular, for a given level of government education expenditure, I examine how resources should be split across k-12 and college education to maximize growth. I then consider the consequences of increasing the level of expenditure given that a constant share subsidizes college.

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6While evidence of the importance of education and human capital in determining growth in observed economies is inconclusive, some evidence suggests it can be quite important. See Temple (1999, 2000) for a discussion of this issue.

7Glomm and Ravikumar (1992) find that public funding may lead to lower per capita income than private funding of education. Zhang (1996) confirms this and shows that subsidies to private education can dominate public education in terms of growth and welfare. Brauninger and Vidal (2000) argue that while a marginal increment to subsidies can decrease growth, growth is maximized with pure public education.

8As discussed later, the extent of this influence is debated.

9It is common to model the labor of agents with different education levels as distinct inputs. For example, see Katz and Murphy (1992), Krusell et. al. (2000), and Blankenau and Ingram (2001).
The framework for analysis is an overlapping generations model where human capital accumulation fuels long run growth. The model is sufficiently stylized to allow analytical results. Young agents make a compulsory investment in k-12 education resulting in a human capital endowment. They also decide whether to make an additional investment in college education. K-12 education is publicly funded; college education requires private expenditures but may be subsidized.\(^{10}\) Agents supply labor inputs in proportion to their human capital endowment. Those with college education additionally supply skill as a separate input to production. The growth rate depends upon quality in k-12 education and the level of skill employed. K-12 quality is proportional to spending and college attendance responds to subsidies. Thus government influences growth both by choosing a level of expenditures and by allocating resources across these uses.

I find that when the level of government education spending is sufficiently small, all resources should provide k-12 quality. When the level rises above a critical value, a positive share should subsidize college if skill and labor are substitutes in production.\(^{11}\) An increase in the share of resources subsidizing college education increases the share of the population earning degrees. This has a positive growth effect. With total expenditures held constant, the requisite decrease in k-12 funding has a negative growth effect. When total education expenditures are small, the negative effect dominates. With a higher level of expenditures, growth is maximized where the two effects offset. In this case, the appropriate share devoted to subsidies is larger when the level of expenditures is larger. Also, subsidies should comprise a smaller share of expenditures when k-12 education is more important in generating human capital and when skill and labor are more substitutable in production.

The growth effect of increasing the level of public education spending depends upon its allocation. Increased spending increases the quality of k-12 education; a

\(^{10}\)There are several papers in the growth literature where both public and private expenditures are inputs in human capital production. See Glomm and Ravikumar (1992, 1998), Cassou and Lansing (2001). Often the private investment is a time cost rather than a tuition cost as in this paper.

\(^{11}\)Specifically, when the elasticity of substitution in the CES class of production functions is greater than 1.
positive growth effect. It may also alter the quantity of college education. The direction of the second effect is unclear. Increased spending directly lowers the private cost of skill but the resulting tax increase crowds out savings, increasing the cost of borrowing and consequently increasing the cost of skill. With expenditures allocated in large part to subsidies, the first effect dominates and increased expenditures increase college enrollment. This is an additional positive growth effect so increasing expenditures increases growth. If a small share is devoted to subsidies, the second effect dominates and college education falls, generating a negative growth effect. In this case, there may be an interior growth maximizing level of expenditures.

This analysis shares some features with Kaganovich and Zilcha (1999). They consider the case where government can provide either of two inputs to human capital. Government is the exclusive provider of the first input while the second can be provided either by government through education vouchers or privately through parental expenditure. Government expenditure on the second input discourages private expenditure by providing precisely the input that parents provide. Also, government expenditure does not influence enrollment. They find that vouchers can increase growth when parental expenditure is low due to a low degree of altruism or when both altruism and total government education expenditures are high. In this paper government expenditure on the privately funded input encourages private investment and influences enrollment. Investment in the private input is made by current learners and altruism is not a factor. Thus while the results in the two papers are complementary, they analyze distinct cases of a government choosing across two types of education expenditures.

The following section describes the environment in which these experiments are conducted. Section 3 describes the possible equilibria in the model and Section 4 looks at policy implications. The final section summarizes and concludes.
2 The Environment.

The economy consists of overlapping generations of three-period-lived, ex-ante homogeneous agents, a government and a single representative firm. The size of each generation is normalized to one. Agents in their first period are learners. Learners are presented sequentially with two education opportunities. The first is compulsory and funded by government. This investment endows each period \( t \) learner with \( h_t \) units of human capital. The second is optional, may be funded in part by government and augments human capital in a manner made explicit below. For brevity, these investments are referred to as k-12 and college education.

2.1 Private education costs and benefits.

Education expenditures for learner \( i \) in period \( t \) depend upon the education level chosen \((e_t^i)\). As all learners make the first investment, agents are differentiated by whether they have a college education \((e_t^i = c)\) or no college education \((e_t^i = n)\). Let \( C_t^i|e_t^i, e_t^i \in \{c, n\} \), be private education expenditures for period \( t \) learner \( i \). Then

\[
C_t^i|e_t^i = \begin{cases} 
0 & \text{if } e_t^i = n \\
T_t (1 - \psi_t) & \text{if } e_t^i = c 
\end{cases}
\]

where \( T_t \) is the tuition cost of a college education and \( \psi_t \in [0, 1] \) is the share paid by government. Learners have neither income nor additional expenses and thus incur debt obligations in this amount.

In the second period, agents are earners. Each inelastically supplies \( h_t \) units of human capital in a competitive labor market. The time \( t \) education-contingent payment per unit of human capital for agent \( i \) is \( \omega_t^i|e_t^i \) where

\[
\omega_t^i|e_t^i = \begin{cases} 
\omega_{n,t+1} & \text{if } e_t^i = n \\
\omega_{n,t+1} + \omega_{c,t+1} & \text{if } e_t^i = c 
\end{cases}
\]

Gross labor income, then, is \( h_t \omega_t^i|e_t^i \) and the college education premium is \( h_t \omega_{c,t+1} \). Earners use income to repay any college loan obligations and to pay taxes. The remainder is allocated across current consumption and savings.

In the third period, agents are old and use income from savings to finance consumption. They have no additional income or expenditures. Each initial earner is
endowed with $h_o$ units of human capital and a fraction, $\Pi_o \in (0, 1]$, of these earners are endowed with both a college education and loan obligations. If $\Pi_o \in (0, 1)$, loan obligations are such that net income is equal across the original earners.

2.2 The agents’ problem.

Generations are indexed by the year that members are learners. Members of generation $t$ are learners in period $t$, earners in period $t + 1$ and old in period $t + 2$. Each member of each generation makes two decisions: an education decision as a learner and a resource allocation decision as an earner.

**Optimal education.** Uncertainty regarding education is resolved prior to resource allocation and a bond market operates freely. Thus the optimal education choice maximizes the present discounted value of lifetime income net of taxes and education expenses. Learners choose an education strategy taking the strategy of others as given and correctly evaluating the returns to each possible outcome. Specifically, in period $t$ learner $i$ of generation $t$ chooses a probability with which to earn a college education, $\pi^i_t \in [0, 1]$, taking the population proportion of college educated workers, $\Pi_j \in (0, 1]$, $\forall j \geq 0$, the cost of education, the education premium and taxes as given.\footnote{There is never an equilibrium with $\Pi = 0$. Intuitively, as $\Pi$ approaches, the wage rate for skilled labor in the following period approaches infinity In this case $\Pi = 0$ cannot be a best strategy. For brevity, I omit this case in the ensuing discussion.}

Taxes are lump sum and do not directly influence education choices. Optimality in education requires

$$
\pi^i_t \begin{cases} 
= 1 & \text{if } h_t \omega_{c,t+1} \geq (1 + r_{t+1})(1 - \psi_t)T_t \\
\in [0, 1) & \text{if } h_t \omega_{c,t+1} = (1 + r_{t+1})(1 - \psi_t)T_t \\
= 0 & \text{if } h_t \omega_{c,t+1} \leq (1 + r_{t+1})(1 - \psi_t)T_t 
\end{cases}
$$

Equation 1 states that an individual prefers to acquire a college education if the present value of the college education premium exceeds the private cost of college.\footnote{It is common in the growth literature to impose a time cost or both a time cost and a goods cost to education. Galor and Moav (2000) and Brauninger and Vidal (2000), however, have only a goods cost proportional to the cost of skill as in this paper. The goods cost in the current model allows tractability. The type of cost is important in papers considering the implications of different taxing schemes (see Milesi-Ferretti and Roubini) since a time input goes untaxed. Here taxes are lump sum which diminishes the importance of the distinction.}
If the premium just compensates the cost, the individual is indifferent and randomizes over the choices. If the cost exceeds the premium, the agent will not earn a degree.

Optimal consumption. Preferences are identical across agents and logarithmic over consumption as an earner and while old with discount rate $\beta$. Let $c_{t,t+1}$ and $c_{t,t+2}$ be consumption by agent $i$ of the generation $t$ in periods $t+1$ and $t+2$. Agent specific indexation is suppressed here to limit notational complexity. A lump sum tax $\tau_t$ is levied on earners. An agent saves by purchasing bonds and bond holdings by the agent in period $t$ are $b_{t,t}$.

Subsequent to the education decision, each agents chooses $b_{t,t+1}$, $c_{t,t+1}$ and $c_{t,t+2}$ to solve the following problem:

$$\max \ln c_{t,t+1} + \beta \ln c_{t,t+2}$$ (2)

$$\text{st} \left\{ \begin{array}{l}
c_{t,t+1} \leq h_{t+1}\omega_{t+1}|e_t - (1 + r_{t+1})C_t|e_t - b_{t,t+1} - \tau_t \\
c_{t,t+2} \leq (1 + r_{t+2})b_{t,t+1}
\end{array} \right\}$$

Solving 2 gives the optimal savings for each earner:

$$b_{t,t+1} = \tilde{\beta} (h_{t}\omega_{t+1}|e_t - (1 + r_{t+1})C_t|e_t - \tau_t)$$ (3)

where $\tilde{\beta} \equiv \frac{\beta}{1+\beta}$.

2.3 The firm’s problem.

A representative firm hires human capital to produce a final output good which is sold in a competitive market. Human capital provides two inputs to production: skill and labor. The quantities of skill and labor employed in period $t$ are $S_t$ and $L_t$. The firm’s production function is

$$Y_t = A \left[(1 - \gamma) S_t^\rho + \gamma L_t^\rho\right]^\frac{1}{\rho}$$

where $A > 0$, $\rho \leq 1$ and $0 \leq \gamma \leq 1$. In the case where $\rho = 0$, $Y_t = AS_t^{(1-\gamma)}L_t^\gamma$. With this specification, $(1 - \rho)^{-1}$ is the elasticity of substitution between skill and labor and $\gamma$ and scales their relative importance. The factor market is competitive so that marginal productivities determine factor prices.
2.4 Human capital.

As in Galor and Moav (2000), the cost of college education is proportional to the cost
of employing skill. Thus

\[ T_t = \theta h_{t-1} \omega_{c,t} \]  

where \( \theta > 0 \) is a scalar. All agents provide labor in proportion to their human capital
endowment. Agents with college education additionally provide skill in proportion to
their human capital endowment. Setting the scale factors to one, the supply of labor
and skill are

\[ L_{t+1} = h_t \]
\[ S_{t+1} = \Pi_t h_t. \]

That is, the labor input depends only on the human capital endowment while the
skill input depends on both the endowment and the private college education choice.\textsuperscript{14}

This specification is particularly convenient in that private decisions effect only the
supply of skill. Given this, and given that the labor market is competitive, when all
human capital is employed wages for skill and labor will be given by

\begin{align*}
\omega_{c,t} &= \frac{Y_t (1-\gamma) \Pi_t^{\alpha-1}}{h_t \left( 1 - \gamma \right) \Pi_t + \gamma} \\
\omega_{n,t} &= \frac{Y_t}{h_t \left( 1 - \gamma \right) \Pi_t + \gamma}.
\end{align*}

Government education expenditures, \( E_t \), convert one for one to units of education
quality. Per capita human capital accumulates according to

\[ h_{t+1} = H (E_t, h_t, \Pi_t) \]

The human capital endowment to the current generation may increase as more re-
sources are directly devoted to its production, (as \( E_t \) increases), as the store of hu-
man capital increases and/or as the number of people employed in skilled positions
increases.

\textsuperscript{14} An alternative to this specification is to have college educated workers provide only skill and
ek-12 educated provide labor. In this case the model is intractable. Numerical exercises show that
the qualitative results are similar under two specifications. It is unclear which best matches observed
economies. The current specification captures that jobs which require a college education include
many tasks for which such training is not needed.
Including the prior generation’s skill level and education level is natural in light of studies that show a strong relationship between education outcomes of children and parents. Including public k-12 expenditures is perhaps more suspect. There is some empirical support for its inclusion. For example Card and Krueger (1992) find evidence that education quality, measured in terms of length of school year, class size and teacher’s salaries positively influences wage rates. More recently Krueger (1998, page 30) argues that “...the widely held belief ...that additional resources yield no benefits in the current system...is not supported in the data.” However, this assessment is not uncontested. For example, Hoxby (2000) finds no evidence that smaller class size increases student achievement. Hanushek (1998, page 12) concludes that “...the United States has made steady and large investments in human capital. The resources invested, however, have had little payoff in terms of student performance.” In light of this uncertainty, k-12 expenditures are included with a parameter to gauge their relative importance in producing human capital.

A balanced growth path potentially exists for any specification of \( H \) that is constant returns to scale in the sole reproducible input, \( h_t \). I consider here only the case where

\[
\begin{align*}
    h_{t+1} &= h_t + BE^H_t [(1 - \alpha) S^\alpha_t + \alpha L^\alpha_t]^{1-\mu} \sigma \neq 0 \\
    h_{t+1} &= h_t + BE^H_t (S_t^{1-\alpha} L_t^{\alpha})^{1-\mu} \sigma = 0
\end{align*}
\]

with parameter restrictions \( B \geq 0 \) and \( \mu, \alpha \in [0,1] \) and \( \sigma \leq 1 \). Here \( \mu \) determines the relative importance of k-12 quality and the human capital aggregate in producing human capital. In the case \( \alpha = 1 \), this is similar to the human capital accumulation equation in Glomm and Ravikumar (1997). If \( B = 0 \), this is a simplified version of the Lucas (1990) specification.

\[\text{Footnotes:} \]
\[\text{15} \text{See Coleman (1966) and Hanushek (1986) for example.} \]
\[\text{16} \text{These papers are part of a much larger discussion on the expenditure/quality relationship in k-12 education. Some evidence is summarized in the two papers. Two particularly interesting additional studies are: Berliner and Biddle (1995) who argue that demographic changes rather than school failure largely explain unimpressive trends in test scores and Hamushek and Kimko (2000) who argue that evidence supports that education quality matters for growth but that expenditures are not linked to meaningful measures of quality.} \]
2.5 Government.

Government is subject to a period balanced budget constraint. Revenue in period \( t \) comes from lump sum taxes on earners and is a fixed as a share, \( g \), of total output. Specifically

\[
\tau_t = g Y_t. \tag{9}
\]

Revenue is split across two uses: purchases of k-12 education quality and college tuition subsidies. The amount spent on quality is \( E_t \). The amount spent on subsidies is equal to the cost of tuition scaled by government’s share and the fraction of learners making the investment; i.e. the amount spent on subsidies in period \( t \) is \( \psi T_t \Pi_t \) where \( \psi \in [0,1] \). Government spending and revenue must satisfy \( E_t + \psi T_t \Pi_t = g Y_t \).

I focus on two aspects of government policy: the effects of changes in the quantity of government education expenditures governed by \( g \), and the effects of changing the mix of government education revenue across these two uses. To facilitate this second investigation, define \( \phi \in [0,1] \) as the share of total revenue spent on subsidies to higher education. Government policy then is a choice of \( \phi \) and \( g \) subject to

\[
E_t = (1 - \phi) g Y_t \tag{10}
\]

\[
\psi T_t \Pi_t = \phi g Y_t. \tag{11}
\]

3 Equilibria.

**Definition 1**  A **competitive Nash equilibrium** in this economy is a sequences of probabilities \( \{\pi^i_t\}_t=0^\infty \), allocations \( \{c^i_{t,t}, c^i_{t,t+1}\}_t=0^\infty \) and bond holdings \( \{b^i_{t,t+1}\}_t=0^\infty \) chosen by each agent \( i_t \in [0,1], \forall t \geq 0 \); population proportions \( \{\Pi_t\}_t=0^\infty \), human capital stocks \( \{h_t\}_t=0^\infty \), prices \( \{r_t, \omega_{c,t}, \omega_{n,t}\}_t=0^\infty \), and fiscal policy choices \( g, \phi, \psi, \{\tau_t, E_t\}_t=0^\infty \) such that: each agent in each generation chooses \( \pi^i_t \) to satisfy equation 1 and \( \pi^i_t = \Pi_t \) \( \forall i,t \); each agent solves equation 2 to determine the optimal consumption pattern, the firm’s choice of labor and capital inputs maximizes profits with prices given by equation 6; agents supply inputs according to equation 5; human capital accumulates...
according to equation 8; government expenditures satisfy equations 9, 10, and 11; the bond market clears (savings equals private education expenditures); factor markets clear; and the good market clears, i.e. $Y_t = c_{t,t+1} + c_{t-1,t+1} + T,\Pi_t$.

Given an equilibrium sequence of probabilities, $\{\Pi_t\}_{t=0}^{\infty}$, and an initial level of human capital, $h_0$, other endogenous values are uniquely determined. Because of this, equilibria are discussed in terms of $\{\Pi_t\}_{t=0}^{\infty}$. In any period, there are two potential types of equilibria: a pure strategy equilibrium with $\pi^i_t = \Pi_t = 1$ and a mixed strategy equilibrium with $\pi^i_t = \Pi_t \in (0,1)$. From equation 1, a mixed strategy equilibrium in period $t$ requires that the return to college education just equals the cost. Earners with and without college degrees, then, have the same lifetime income net of education costs and consequently save the same amount. From equation 3, this amount is $\bar{\beta} (h_{t-\omega,t} - \tau_t)$.

Bond market clearing requires that savings by earners equals borrowing by learners. With $\Pi_t \in (0,1)$ this is:

$$\bar{\beta} (h_{t-\omega,t} - \tau_t) = \Pi_t \theta (1 - \psi_t) \omega_{c,t} h_{t-1}.$$  

Substituting equations 9 and 11 into this and simplifying gives

$$\bar{\beta} (h_{t-\omega,t} - gY_t) + \phi gY_t = \Pi_t \omega_{c,t} h_{t-1}. \quad (12)$$

The first expression on the left hand side is period $t$ savings and the second is government college education expenditures. Bond market clearing requires that the sum of these equal total college expenditures (right hand side). Substituting in for wages, dividing each side by $Y_t$ and solving for $\Pi_t$ yields the following difference equation in $\Pi_t$:

$$\Pi_t = \Pi_{t-1}^{1-\rho} \frac{\gamma}{1-\gamma} \left( \bar{\beta} \theta + \frac{g}{\theta} (\phi - \bar{\beta}) \right) + \frac{g}{\theta} (\phi - \bar{\beta}) \Pi_{t-1}. \quad (13)$$

**Definition 2** A balanced growth path exists when output and human capital grow at a common, constant rate and the share of the population earning degrees is constant at some $\Pi \in (0,1]$. 

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(i) Along a mixed strategy balanced growth path \( \Pi \in (0, 1) \).

(ii) Along a pure strategy balanced growth path \( \Pi = 1 \).

Letting \( \Pi_t = \Pi_{t+1} = \Pi^* \) in equation 13 gives the value of \( \Pi \) along a mixed strategy balanced growth path:

\[
\Pi^* = \left( \frac{\gamma \tilde{\beta} + g \left( \phi - \tilde{\beta} \right)}{1 - \gamma \theta - g \left( \phi - \tilde{\beta} \right)} \right)^{\frac{1}{\rho}}
\]  
(14)

Proposition 1 gives conditions under which pure and mixed strategy balanced growth paths exist and under which there is convergence to a balanced growth path. All proofs are in the appendix.

**Proposition 1** Define \( \phi_c \equiv \frac{\theta}{g} + \tilde{\beta} - \frac{\gamma}{g} \left( \tilde{\beta} + \theta \right) \).

(i) When \( \rho > 0 \) and \( \phi \leq \phi_c \), both a pure and mixed strategy balanced growth path exist. If education is sufficiently costly, the economy converges locally to the mixed strategy balanced growth path. The pure strategy balanced growth path is supported only if the economy begins along this path.

(ii) When \( \rho < 0 \), a pure strategy balanced growth path exists if \( \phi \leq \phi_c \) and a mixed strategy balanced growth path exists if \( \phi_c \leq \phi < \frac{\theta}{g} + \tilde{\beta} \). In each case, the balanced growth path is supported only if the economy begins along this path.

(iii) When \( \rho = 0 \), a balanced growth path exists only if \( \phi = \phi_c \). The level of \( \Pi \) is indeterminate.

The sign of \( \rho \) dictates the effect of education subsidies on the level of college education. Empirical evidence suggests that \( \rho > 0 \) so I focus on this case in the following discussion.\(^{17}\) For contrast and completeness, I briefly discuss the other cases. Corollary 1 summarizes conditions under which a mixed strategy balanced growth path exists with \( \rho > 0 \).

**Corollary 1** Let \( \rho > 0 \). If \( g > \min \left[ g_c \equiv \gamma - \frac{\theta}{\beta} (1 - \gamma), 0 \right] \) a mixed strategy balanced

\(^{17}\)Katz and Murphy (1992) estimate \( \rho \approx .29 \). Blankenau (1999) estimates \( \rho \approx .414 \).
growth path exists for some $\phi > 0$. If in addition $g < g_a \equiv \frac{\theta - \gamma(\beta + \theta)}{(1 - \beta)}$, a mixed strategy balanced growth path exists for any $\phi$.

The requirement that $g$ exceed $g_c$ is a requirement that borrowing exceed savings when all agents attend college and tuition is not subsidized; i.e. when $\Pi = 1$ and $\phi = 0$. To see this, divide each side of 12 by $Y_t$. Dropping time subscripts and substituting for wages, this gives:

$$\bar{\beta} \left( \frac{\gamma}{[(1 - \gamma) \Pi^\rho + \gamma]} - g \right) + g\phi = \theta (1 - \gamma) \frac{\Pi^\rho}{[(1 - \gamma) \Pi^\rho + \gamma]}$$

(15)

Setting $\phi = 0$ and $\Pi = 1$ in equation 15 gives $\bar{\beta} (\gamma - g) = \theta (1 - \gamma)$ which is equivalent to $g = g_c$. If $g$ decreases from $g_c$ with $\phi = 0$, the left-hand side in equation 15 is larger. To preserve the equality, the right-hand side must increase or the other left-hand side term must decrease. With $\rho > 0$, either of these requires an increase in $\Pi$. But $\Pi > 1$ is not possible. Thus $g > g_c$ does not allow bond market clearing.18 Intuitively, when $\phi = 0$ and $\Pi = 1$ there are no subsidies and all agents borrow for college. Borrowing is as high as possible. If savings exceed borrowing in this case, the bond market will not clear for any parameterization.

As $\phi$ increases from 0, the left hand side of equation 15 decreases and an increase in $\Pi$ is required to preserve the inequality. It is straightforward to show that with $g > g_c$ and $\phi = 0$, $\Pi^* < 1$. Thus over some range of $\phi$, an increase in $\Pi$ is possible. As $\phi$ increases further, eventually either $\phi = 1$ or $\Pi^* = 1$. If $\phi = 1$, further increases are not possible. If $\Pi^* = 1$, further increases in $\phi$ cannot generate further increases in $\Pi^*$. Thus the equality cannot hold and a mixed strategy balanced growth path does not exist. When $g < g_a$, $\Pi^* < 1$ even with $\phi = 1$ so that such a growth path can always exist.

The parameter space that supports a mixed strategy equilibrium also supports a pure strategy equilibrium with $\Pi = 1$. To see why, note that with $\Pi = 1$ bond market clearing requires

$$\bar{\beta} (\gamma - g + z) = \theta (1 - \gamma) - g\phi.$$  

(16)

18With $\rho < 0$, preserving the equality subsequent to a decrease in $g$ from $g_c$ requires a decrease in $\Pi$. This is possible with $\Pi = 1$ so $g < g_c$ allows bond market clearing.
where $z$ is the net return to schooling; $z \equiv h_{t-1} \omega_c - (1 + r_t) (1 - \psi) \theta \omega_c h_{t-2}$. The left hand side is savings (see equation 3) and the right hand side is private education expenditures with $\Pi = 1$. In order for $\Pi = 1$ to be optimal, it must be that $z \geq 0$.\footnote{As $\rho$ is not a factor in this argument, an equilibrium with $\Pi = 1$ is also supported with $\phi \leq \phi_c$ for $\rho < 0$.} Solving for $z$ in 16, it is straightforward to show $z \geq 0$ requires $\phi \leq \phi_c$.\footnote{When subsidies increase with $\Pi = 1$, there is less borrowing. To preserve the equality, savings must fall. It can be shown that $r_t$ adjusts to preserve bond market clearing. This increases loan repayments, reducing savings.} Proposition 1 shows that while both exist, convergence is possible only to the empirically relevant mixed strategy equilibrium.

4 Policy Analysis

4.1 The effects of policy on education levels.

Consider the effect of government policy on the level of college education along the balanced growth path. Along the pure strategy balanced growth path there is clearly no effect. Corollary 2, which follows from equation 14, summarizes the effect along the mixed strategy balanced growth path.

**Corollary 2.** Consider the parameter space for which a mixed strategy balanced growth path exists. When $\rho > 0$ ($\rho < 0$)

(i) $\Pi^*$ is positively (negatively) related to $\phi$.

(ii) $\Pi^*$ is positively (negatively) related to $g$ if $\phi > \bar{\beta}$

(iii) $\Pi^*$ is negatively (positively) related to $g$ if $\phi < \bar{\beta}$.

Increasing the share of funding going to college education increases enrollment only if $\rho > 0$ (item (i) above). The effect of increasing total expenditures depends on how the funds are used. To understand items (ii) and (iii) it is convenient to rewrite equation 15 as

$$g \left( \phi - \bar{\beta} \right) = \theta \frac{(1 - \gamma) \Pi^\rho}{\left[ (1 - \gamma) \Pi^\rho + \gamma \right]} - \frac{\gamma \bar{\beta}}{\left[ (1 - \gamma) \Pi^\rho + \gamma \right]}.$$  (17)

The left hand side of this expression is the net direct effect of government policy.
in the bond market. Increasing $g$ by one unit increases public funding of college education by $\phi$ per unit of output. However, the increase in $g$ is funded by an equal increment to the per capita tax burden of earners. Earners decrease their savings by $\beta$ per unit of output in response.\footnote{In part, this is due to the tax structure. Only earners in this model are savers and only earners are taxed. Thus an increase in taxation, absent general equilibrium adjustment decreases savings. Collecting some taxes from the old would diminish this effect as it does in Uhlig and Yanagawa (1996) and Blankenau and Ingram (2001). However, taxing only the old is not realistic and the basic argument holds when some tax is imposed on learners. In this model, taxing the old greatly increases the notational complexity and yields no appreciably different results.} If $\phi > \beta$, the increased supply of government funds outweighs the decreased supply of private funds and the net direct impact is an increase in funds available for college education. Available funds increase also if $\phi$ increases as this requires no change in the tax rate but increases government college education spending.

Bond market clearing requires an indirect effect through changes in $\Pi$ to offset the increase. These indirect effects are captured on the right-hand side of equation 17. The first expression is total education expenditures as a fraction of output $(\text{derived from } \frac{\Pi \omega_h h_t}{Y_t})$. As $\Pi$ increases, more agents borrow for education. However, the resulting decrease in skilled wages lowers tuition costs and per student borrowing falls. When $\rho > 0$, the first effect dominates and the education expenditures ratio is increasing in $\Pi$. The second expression is savings as a fraction of output given zero taxes $(\text{derived from } \frac{\beta \omega_n h_t}{Y_t})$. With $\rho > 0$, an increase in skill decreases $\omega_n$ relative to output. As savings is proportional to $\omega_n$, the savings rate is decreasing in $\Pi$. As this enters negatively in the expression, the right hand side of 17 is increasing in $\Pi$. Thus a government policy which increases available funds requires an increase in $\Pi$ to maintain bond market clearing. Similar reasoning reveals that with $\rho < 0$, a decrease in $\Pi$ is required.

\textbf{4.2 The growth implications of k-12 versus college public education expenditures.}

The primary purpose of this paper is to evaluate the implications of government policy choices for long run growth. Let $\lambda_{t+1} \equiv \frac{h_{t+1} - h_t}{h_t}$ be the growth rate of human
capital in period $t + 1$. Then substituting for $E_t$, $S_t$ and $L_t$ and dividing both sides of equation 8 by $h_t$ yields the following equation for period $t + 1$ growth:

$$\lambda_{t+1} = \left((1 - \phi) g A \left[(1 - \gamma) \Pi_t^\rho + \gamma\right]^{\frac{1}{\rho}}\right) \mu \left[(1 - \alpha) \Pi_{t-1}^\rho + \alpha\right]^{\frac{1-\mu}{\rho}}.$$

Note that when $0 < \mu < 1$, $\Pi$ influences growth both directly and indirectly through its impact on output. Analytical results are available in the special case where $\sigma = \rho$ and $\alpha = \gamma$. Thus the growth implications of k-12 versus college public education expenditures are first discussed assuming these parameter restrictions and a subsequent numerical exercise shows sensitivity of the findings to changes in $\sigma$. With $\sigma = \rho$ and $\alpha = \gamma$, $h_t$ and $\Pi_t$ enter the final goods and human capital accumulation functions symmetrically. Simplifying gives the following expression for the growth rate of human capital along the balanced growth path:

$$\lambda = A^\mu ((1 - \phi) g)^{\mu} [(1 - \gamma) \Pi^\rho + \gamma]^{\frac{1}{\rho}}. \quad (18)$$

A change in $\phi$ potentially affects the growth rate in two ways: directly through its influence on the human capital endowment and indirectly through its effect on $\Pi$. Along a mixed strategy balanced growth path with $\rho > 0$, these are competing effects. An increase in $\phi$ decreases k-12 quality but increases the share of the population acquiring skill. This opens the possibility that subsidies can increase growth. In contrast, with $\rho < 0$ an increase in $\phi$ decreases $\Pi$ and with $\Pi = 1$ the indirect effect is not operative. In each of these cases subsidies necessarily lower growth. Proposition 2 summarizes the share of resources that should subsidize college education to maximize growth in each of these circumstances.

**Proposition 2** Let $\sigma = \rho$, $\alpha = \gamma$ and define $\phi^* \equiv \frac{1-\mu \rho (\frac{g}{\gamma} + \beta)}{1-\mu \rho}$. If $\rho > 0$ and $0 \leq \phi^* \leq \phi_c$ the mixed strategy balanced growth rate is maximized at $\phi = \max [\phi^*, 0]$. Otherwise, the balanced growth rate is maximized at $\phi = 0$.

In the proof to proposition 2 it is shown that $\phi < \phi_c$ is sufficient for $\frac{\theta}{g} + \frac{\beta}{\gamma} > 1$ so that $\phi^* < 1$ is assured. Using the envelope theorem this also assures $\frac{\partial \phi^*}{\partial \rho}, \frac{\partial \phi^*}{\partial \mu} < 0$. 

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Corollary 3 summarizes features of the growth maximizing share of expenditures allocated to college subsidies.

**Corollary 3** It is never growth maximizing to allocate all resources to subsidies. It is growth maximizing to devote no resources to subsidies (i) along a pure strategy balanced growth path, (ii) along a mixed strategy balanced growth path with $\rho < 1$ and (iii) along a mixed strategy balanced growth path with $\rho > 0$ and $g < \frac{\theta \mu \rho}{1 - \beta \mu \rho}$.

Along a mixed strategy balanced growth path with $\rho > 0$ and $g > \frac{\theta \mu \rho}{1 - \beta \mu \rho}$ some resources should be devoted to college education. The growth maximizing share is larger when total education expenditures are larger, when skill is more important in producing human capital, and when skill and labor are more complementary in production.

Since growth is necessarily zero when all resources are devoted to subsidies, it is not surprising that this is never growth maximizing. When $\Pi = 1$ subsidies cannot encourage further college enrollment and when $\rho < 1$ subsidies lower college enrollment along a mixed strategy balanced growth path. In these cases, taking any resources from education quality to subsidizes college education will lower growth.

To understand why it is optimal to not subsidize college when $g$ is small (item iii above), put $\Pi^*$ into 18 and rearrange to obtain the following expression for growth

$$\lambda = ((1 - \phi) gA)^\mu \left[ \frac{\gamma \left( \beta + \theta \right)}{\theta - g \left( \phi - \bar{\beta} \right)} \right]^{\frac{1}{\rho}}. \quad (19)$$

This expression highlights that an increase in $\phi$ has two growth effects. First it lowers the share of income going to education quality. The growth effects of this are captured in the expression $((1 - \phi) gA)^\mu$. Secondly, it increases growth through an increase in $\Pi$. The effects of this are captured in the bracketed part of expression 19. Whether an increase in $\phi$ increases or decreases growth depends on the relative size of these effects. It is straightforward to show that the elasticity of growth with respect to $\phi$
change in $\phi$ ($\varepsilon_{\lambda\phi} \equiv \frac{\partial \lambda}{\partial \phi}$) is equal to

$$
\varepsilon_{\lambda\phi} = -\frac{\mu\phi}{(1 - \phi)} + \frac{g\phi}{\rho \left( \theta - g \left( \phi - \beta \right) \right)}
$$

The first right-hand side expression is negative and measures the responsiveness of growth from changes in the share of output devoted to quality when $\phi$ increases. This is referred to as the “share effect” for brevity. The second is positive with $\rho > 0$ and measures the responsiveness of growth from changes in $\Pi$ when $\phi$ increases.\(^{22}\) This is referred to as the “skill effect.” Growth can be increased by increasing $\phi$ so long as the share effect is smaller than the skill effect in absolute value; i.e. so long as $\varepsilon_{\lambda\phi} > 0$ or

$$
\frac{g}{\left( \theta - g \left( \phi - \beta \right) \right)} \geq \frac{\mu}{(1 - \phi)}
$$

At $\phi = 0$, this requires $g > \frac{\theta \mu \rho}{1 - \beta \mu \rho}$. Thus when $g$ is sufficiently small, growth cannot be increased with subsidies.

Expression 20 as an equality is also useful in understanding why the growth maximizing value of $\phi$, when not equal to zero, is increasing in $g$. Notice that the share effect is unchanging in $g$ while the skill effect is increasing in $g$. Thus $\phi$ must increase with $g$ to preserve the equality. More intuitively as $g$ increases and education quality becomes large, the growth effect of additional skill becomes large relative to the effect of additional quality.

An increase in $\mu$ increases the importance of quality in generating growth. As a consequence the share effect increases in absolute value with $\mu$. The more important is quality in generating growth, the larger the effect of taking resources from k-12 quality for subsidies. Thus to maximize growth a greater share of resources should finance k-12 quality; college subsidies should be lower.

Notice also that an increase in $\rho$ decreases the skill effect. When $\rho$ is larger, labor is a better substitute for skill in production. This has two complementary consequences for the effect of subsidies on growth. First, the responsiveness of $\Pi^*$ to

\(^{22}\theta - g \left( \phi - \beta \right) > 0 \text{ when } \phi < \phi_c \text{ as required.}
a change in \( \phi \) is smaller. \(^{23}\) Secondly, when \( \rho \) is larger, growth is less responsive to an increase in \( \Pi \). \(^{24}\) With \( \Pi^* \) less responsive to \( \phi \) and growth less responsive to skill, the growth effect of increasing subsidies is smaller; college subsidies should be lower when \( \rho \) is larger.

### 4.3 The growth implications of aggregate public education expenditures.

Consider now the effect of increasing \( g \) holding \( \phi \) fixed. Recall that with \( \rho > 0 \), \( g \) and \( \Pi^* \) are positively related if \( \phi > \bar{\beta} \). In this case, increasing \( g \) will increase growth since both the supply of skill and the quantity of education quality are larger. The same is true when \( \rho < 0 \) and \( \phi < \bar{\beta} \). If instead \( \rho > 0 \) and \( \phi < \bar{\beta} \), increasing \( g \) has competing effects on growth: there is a direct increase in the share of output devoted to education quality but \( \Pi \) falls which, on its own, would decrease growth. This holds also when \( \rho < 0 \) and \( \phi > \bar{\beta} \). In these instances, the growth effects of public education expenditures are not immediately clear. Proposition 3 summarizes these effects.

**Proposition 3** Define \( g^* \equiv -\frac{\theta}{(\phi - \bar{\beta})\frac{\mu \rho}{1 - \mu \rho}} \). If \( \rho > 0 \) and \( 0 \leq \phi \leq \min\{\phi_c, \bar{\beta}\} \), the mixed strategy balanced growth rate is maximized at \( g = g^* \). If \( \rho < 0 \) and \( \max\{\phi_c, \bar{\beta}\} \leq \phi < \frac{\theta}{g} + \bar{\beta} \), the mixed strategy balanced growth rate is maximized at \( g = g^* \). Otherwise, the balanced growth rate is increasing in \( g \).

A nontrivial equilibrium with \( g = g^* \) can exist only if \( g^* > 0 \). With \( \rho > 0 \) this requires \( (\phi - \bar{\beta}) < 0 \). \(^{25}\) In this case, \( g^* \) is increasing in both \( \mu \) and \( \rho \). The growth maximizing \( g \) balances the direct effect of increased k-12 expenditures with the indirect effect of a decrease in \( \Pi \). When \( \mu \) rises, k-12 quality is more important in generating human capital. When \( \rho \) rises, labor and skill are more substitutable in production. Thus increases in these parameters diminish the significance of the decrease in \( \Pi \) from crowding out and increases the growth maximizing level of expenditures.

\(^{23}\) Straightforward calculations yield \( \varepsilon_{\Pi, \phi} \equiv \frac{\Pi^*}{\phi^*} = \frac{\theta + \bar{\beta}}{\theta - \bar{\beta}} g^* \left( \left( \theta - g \left( \phi - \bar{\beta} \right) \right) \left( \bar{\beta} + g \left( \phi - \bar{\beta} \right) \right) \rho \right)^{-1} \) which is decreasing in \( \rho \).

\(^{24}\) Straightforward calculations yield \( \varepsilon_{\Pi, \lambda} \equiv \frac{\partial \Pi}{\partial \Pi} = \frac{(1 - \gamma)\mu^\rho}{\mu^\rho (1 - \gamma)^{\mu^\rho + \gamma}} \) which is decreasing in \( \rho \).

\(^{25}\) With \( \rho < 0 \) this requires \( (\phi - \bar{\beta}) > 0 \).
When $\theta$ rises, education is more costly and the fall in $\Pi$ from an increase is $g$ is smaller. Since the indirect effect is less onerous, $g^*$ is larger. Recall that $\left(\phi - \bar{\beta}\right)$ is the net direct impact of government policy in the bond market. An increase in $\left(\phi - \bar{\beta}\right)$ means that the net drain of resources from the bond market from an increase in $g$ is smaller. With $\rho > 0$, this means that the negative impact on $\Pi$ of $g$ is smaller and $g^*$ is larger.

4.4 Sensitivity analysis

While $\rho = \sigma$ and $\alpha = \gamma$ are required for analytical results, relaxing these assumption does not alter the findings above in a significant way. Figure 1 below shows values of $g^*$ and $\phi^*$ for $\sigma \in [-.5,.5]$ holding $\rho$ and other values constant with $\alpha = \gamma$. Note that as skill and labor become more substitutable in producing human capital (as $\sigma$ increases) the growth maximizing share of expenditures devoted to college education falls and the growth maximizing level of government education expenditures rises. This is consistent with the effects of increasing $\rho$ in the case where $\rho = \sigma$ (see above expressions for $g^*$ and $\phi^*$). Parameters here are chosen such that $g^*, \phi^* \in (0,1)$ but are otherwise arbitrary. Conducting the experiment for a wide variety of parameters shows this relationship to be robust. Thus changes in $\sigma$ alone have the same effect qualitatively as changes in $\rho$ and $\sigma$ simultaneously; the assumption that $\rho = \sigma$ is apparently innocuous in terms of qualitative results. Results are similar in the case where $\rho = \sigma, \alpha \neq \gamma$. In particular $g^*$ is increasing and $\phi^*$ is decreasing in $\alpha$.

The assumption that skill and labor are separate inputs in production is not essential to the results of the paper. The parameter $\rho$ is not an argument in $\phi_c$ and with $\rho = 1$, $\phi^* \equiv \frac{1-\mu(\frac{\theta}{\rho + \theta})}{1-\mu}$ and $g^* \equiv \frac{\theta}{\beta - \phi} \mu \frac{1}{1-\mu}$. Thus so long as $\mu < 1$, nontrivial policy choices may remain. That is, the above analysis nests the more commonly developed case where private education expenditures simply increase the supply of the single labor input. Similarly, if $\rho < 1$ the results are robust to the case where the human capital aggregate is not a separate input in human capital production, i.e. where $\mu = 1$. 

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5 Conclusion.

Government plays an important role in funding both k-12 and college education. Policymakers must choose both a level of education funding and its allocation across these uses. This paper analyzes the effects of both decisions. When skill and labor are complements in production, all resources should be devoted to k-12 education to maximize growth. In the empirically relevant case of substitutability, if total expenditures are large enough, some share of the revenue should go to subsidize college tuition. The share should be larger when total expenditures are larger, when college education is less costly, and when skill and labor are more complementary in production. If a large enough share of public expenditures goes to subsidies, increased total expenditures increase growth. However, if too small a share goes to subsidies, increased spending may decrease both the share of the population attending college and the growth rate. In this case, the growth maximizing level of expenditures is increasing in the share of expenditures devoted to subsidies. Together these results suggest that increased education spending is more likely to increase growth when a
large share is devoted to subsidizing college education.

The model is stylized to allow concise results. As such, additional insights or refinements of the above might be gained from generalizations. The most obvious omission from the model is physical capital. Its inclusion would present an additional tension in each of the main experiments. In the first, if capital is more complementary with skill than labor, increasing subsidies may have a significant secondary effect by also increasing the level of physical capital investment. In the second, increasing total expenditures would have a more complex effect as the required tax increase displaces physical capital. A second meaningful generalization would be a more complex and realistic treatment of taxation. Many papers which consider fiscal policy in endogenous growth models are concerned with the growth implications of distortionary taxes and several authors have shown that the effect of productive expenditures is influenced by how such expenditures are financed. As taxes are lump sum in this paper, such experiments cannot be conducted. With these modifications, the model would lend itself to a careful calibration to gauge the magnitude of the effects of policy changes. However, tractability would be forfeited. This is the direction of future related research.

A Appendix.

Proof of Proposition 1.

A mixed strategy balanced growth path can exist only if \(0 \leq \Pi^* \leq 1\). The numerator of \(\Pi^*\) from equation 14 can be written as \(\gamma \left( \bar{\beta} (1 - g) + g \phi \right)\) which is positive. Thus \(\Pi^* \geq 0\) if the numerator is positive. This reduces to \(\phi < \frac{\theta}{g} + \bar{\beta}\). If \(\rho > 0\), then \(\Pi^* \leq 1\) if the denominator exceeds the numerator. This holds when \(\phi \leq \phi_c\). If \(\rho < 0\), \(\Pi^* \leq 1\) requires that \(\phi \geq \phi_c\). Since \(\phi_c < \frac{\theta}{g} + \bar{\beta}\), with \(\rho > 0\), \(\phi \leq \phi_c\) is sufficient for both \(\Pi^* \geq 0\) and \(\Pi^* \leq 1\). With \(\rho < 0\), \(\phi_c \leq \phi < \frac{\theta}{g} + \bar{\beta}\) is required.

Next consider a pure strategy balanced growth path with \(\Pi^* = 1\). Substituting in for \(\omega_{t+1}^i|c_t^i\) and \(C_{t+1}^i|c_t^i\), equation 3 gives savings. Since this must equal borrowing \((\theta (1 - \bar{\psi}) \omega c_t h_{t-1})\), bond market clearing requires

\[
\bar{\beta} (h_{t-1} \omega_{t,t} + h_{t-1} \omega c_t - (1 + r_t) (1 - \bar{\psi}) \omega c_{t-1} h_{t-2} - \tau_t) = \theta (1 - \bar{\psi}) \omega c_t h_{t-1}.
\]

\(^{26}\)For example, this is a consideration in Glomm and Ravikumar (1998) and Baier and Glomm (2001).
As wages and the interest rates are constant with $\Pi$ constant, these time subscripts are dropped. Note from equations 11 and 4 that with $\Pi_t = 1$, $\psi = \frac{\theta gY_t}{\beta \omega_c}$. Using this and equation 9, the above can be rearranged as

$$(h_{t-1} \omega_c - (1 + r_t)(1 - \psi) \omega_c h_{t-2}) = \frac{\theta \omega_c h_{t-1}}{\beta} - \frac{\phi g Y_t}{\beta} - \omega_c h_{t-1} + g Y_t.$$  \tag{21}

Substituting in for the $T_t$, equation 1 states that $\Pi_t = 1$ is an equilibrium only if $h_{t-1} \omega_c - (1 + r_t)(1 - \psi) \omega_c h_{t-2} \geq 0$. From equation 21 this is equivalent to $\frac{\theta \omega_c h_{t-1}}{\beta} - \frac{\phi g Y_t}{\beta} - \omega_c h_{t-1} + g Y_t \geq 0$ which reduces to $\phi \leq \phi_c$. Thus with $\rho > 0$ and $\phi \leq \phi_c$ both a pure and mixed strategy balanced growth path exist. With $\rho < 0$, a pure strategy balanced growth exists if $\phi \leq \phi_c$ and a mixed strategy balanced growth path exists if $\phi_c \leq \phi < \frac{2}{\beta} + \beta$. From equation 13 it is clear that when $\rho = 0$, a balanced growth path exists only if $\phi = \phi_c$ and the level of $\Pi$ is indeterminate.

Now consider under what conditions $\Pi$ converges locally to $\Pi^*$ when $\Pi_0 \neq \Pi^*$. Write equation 13 as $\Pi_t = f(\Pi_{t-1})$ where $f(\Pi_{t-1}) = a_1 \Pi_{t-1} + a_2 \Pi_{t-1} - 1$ with $a_1 = \frac{\gamma}{\beta} \left( \frac{\beta}{\beta} + \frac{\theta}{\beta} (\phi - \beta) \right)$ and $a_2 = \frac{\theta}{\beta} (\phi - \beta)$. It is straightforward to show that $a_2 \leq 1$ when $\phi \leq \frac{\theta}{\beta} + \beta$ as it must when $\Pi^* \geq 0$. A first order Taylor series expansion gives

$$\Pi_t \approx f(\Pi_{t-1}) \mid_{\Pi_{t-1}=\Pi^*} + f'(\Pi_{t-1}) \mid_{\Pi_{t-1}=\Pi^*} (\Pi_{t-1} - \Pi^*)$$

Note that $f(\Pi_{t-1}) \mid_{\Pi_{t-1}=\Pi^*} = \Pi^*$ and $f'(\Pi_{t-1}) \mid_{\Pi_{t-1}=\Pi^*} = (1 - \rho) a_1 (\Pi^*)^{-\rho} + a_2$. Inspection of equation 14 reveals that $(\Pi^*)^{-\rho} = \frac{1 - a_2}{a_1}$. Thus

$$\Pi_t \approx \Pi^* + ((1 - \rho)(1 - a_2) + a_2) (\Pi_{t-1} - \Pi^*)$$

The behavior of the difference equation is governed by $((1 - \rho)(1 - a_2) + a_2)$ and there is convergence if $|((1 - \rho)(1 - a_2) + a_2)| < 1$. Note $((1 - \rho)(1 - a_2) + a_2) < 1$ holds if and only if $\rho > 0$. $((1 - \rho)(1 - a_2) + a_2) > -1$ holds if and only if $a_2 > \frac{2 - 2}{\rho}$ or $\phi > \frac{\theta - 2\theta}{\rho} + \beta$. If $\theta > g\beta$ this holds for any $\rho, \phi$.

When $\rho = 0$, 13 reduces to $\Pi_t = \frac{\beta \gamma + g(\phi - \beta)}{\theta (1 - \gamma)} \Pi_{t-1}$. This will go to zero or infinity unless $\beta \gamma + g(\phi - \beta) = \theta (1 - \gamma)$ which requires $\phi = \phi_c$. When this holds, any level of $\Pi$ can be supported.

**Proof of proposition 2.** Define $Z \equiv [(1 - \gamma) \Pi^\rho + \gamma]^\frac{1}{\rho}$. From equation 18, then,

$$\frac{\partial \lambda}{\partial \phi} = -\mu g A ((1 - \phi) g A)^{\mu-1} Z + ((1 - \phi) g A)^{\mu} \frac{\partial Z}{\partial \phi}.$$  \tag{22}

Thus in a pure strategy equilibrium with $\Pi = Z = 1$, $\frac{\partial \lambda}{\partial \phi} = -\mu g ((1 - \phi) g A)^{\mu-1} Z < 0$ and $\phi = 0$ maximizes growth. To find a convenient expression for $Z$ in terms of $\phi$, note from equation 14 that $(\Pi^*)^\rho (1 - \gamma) = \frac{\beta + g(\phi - \beta)}{\theta - g(\phi - \beta)}$. Adding $\gamma$ to each side, and raising each to the power $\rho^{-1}$ gives

$$Z = \left( \frac{\gamma (\beta + \theta)}{\theta - g(\phi - \beta)} \right)^{\frac{1}{\rho}}.$$  

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so that

$$\frac{\partial Z}{\partial \phi} = \frac{\rho^{-1}Zg}{\theta - g(\phi - \beta)}.$$  \hspace{1cm} (23)$$

If $\rho < 0$, $\frac{\partial Z}{\partial \phi} < 0$, $\frac{\partial \lambda}{\partial g} < 0$ and $\phi = 0$ maximizes growth (recall $\theta - g(\phi - \beta) > 0$ when a mixed strategy balanced growth path exists). When $\rho > 0$, $\frac{\partial Z}{\partial \phi} > 0$ and growth is maximized where $\frac{\partial \lambda}{\partial g} = 0$. Putting the expression for $\frac{\partial Z}{\partial \phi}$ into $\frac{\partial \lambda}{\partial g} = 0$ and simplifying shows that growth is maximized where $\phi = \phi^*$. To check second order conditions, note $\frac{\partial \lambda}{\partial g} |_{\phi = 0} = Z(gA)^\mu \left( -\mu + \frac{\rho^{-1}g}{\theta + \beta g} \right)$ which is positive if $g > \frac{\theta \mu \rho}{(1-\mu \rho \beta)}$ but this is required for $\phi^* > 0$. Furthermore $\lim_{\phi \to 1} \frac{\partial \lambda}{\partial g} = -\infty$. Since $\frac{\partial \lambda}{\partial g} = 0$ only at $\phi = \phi^*$, $\frac{\partial \lambda}{\partial g}$ is decreasing in $\phi$ and second order conditions are met.

**Proof of proposition 3.**

Define $Z$ as in the previous proof. From equation 18, then,

$$\frac{\partial \lambda}{\partial g} = \mu A (1 - \phi) (g (1 - \phi) A)^{\mu - 1} Z + ((1 - \phi) gA)^\mu \frac{\partial Z}{\partial g}.$$  \hspace{1cm} (24)$$

In a pure strategy equilibrium with $\Pi = Z = 1$, $\frac{\partial \lambda}{\partial g} = \mu (1 - \phi) (g (1 - \phi) A)^{\mu - 1} Z > 0$ and the growth rate is increasing in $g$. When $\Pi = \Pi^*$

$$\frac{\partial Z}{\partial g} = \frac{\rho^{-1}Z(\phi - \beta)}{\theta - g(\phi - \beta)}.$$  \hspace{1cm} (25)$$

When $\rho > 0$ and $\phi > \beta$, $\frac{\partial Z}{\partial g} > 0$ and the balanced growth rate is increasing in $g$. When $\rho > 0$ and $\phi < \beta$, $\frac{\partial Z}{\partial g} < 0$ and growth is maximized where $\frac{\partial \lambda}{\partial g} = 0$. Putting the expression for $\frac{\partial Z}{\partial g}$ into $\frac{\partial \lambda}{\partial g}$ gives

$$\frac{\partial \lambda}{\partial g} = Z(A(1 - \phi))^\mu \left( \mu g^{\mu - 1} + g^\mu \frac{\rho^{-1}(\phi - \beta)}{\theta - g(\phi - \beta)} \right).$$

Setting this equal to 0 and simplifying shows that growth is maximized where $g = g^*$. Since $g$ cannot be less than 0 and $\phi \leq \phi_c$ is required for the existence of an equilibrium with $\rho > 0$ and $\Pi = \Pi^*$, $0 \leq \phi \leq \min \left\{ \phi_c, \beta \right\}$ is required for $g = g^*$ to be the growth maximizing amount of education expenditures.

With $\rho < 0$ and $\phi \leq \beta$, $\frac{\partial Z}{\partial g} > 0$ and the balanced growth rate is increasing in $g$. When $\rho < 0$ and $\phi > \beta$, $\frac{\partial Z}{\partial g} < 0$ and growth is maximized where $\frac{\partial \lambda}{\partial g} = 0$. Putting the expression for $\frac{\partial Z}{\partial g}$ into $\frac{\partial \lambda}{\partial g} = 0$ and simplifying shows that growth is maximized where $g = g^*$. Since $\phi_c \leq \phi < \frac{\theta}{g} + \beta$ is required for the existence of an equilibrium with $\rho < 0$ and $\Pi = \Pi^*$, $\max \left\{ \phi_c, \beta \right\} \leq \phi < \frac{\theta}{g} + \beta$ is required for $g = g^*$ to be the growth maximizing amount of education expenditures.
Consider now the second order conditions.

\[
\frac{\partial^2 \lambda}{\partial g} \Big|_{g=g^*} = \frac{\partial Z}{\partial g} \tilde{A} \left[ \mu (g^*)^{\mu-1} + (g^*)^\mu \frac{\rho^{-1} (\phi - \beta)}{\theta - g^* (\phi - \beta)} \right]
+ Z\tilde{A} (g^*)^\mu \left[ \frac{\mu \rho^{-1} (\phi - \beta)}{g^* \theta - g^* (\phi - \beta)} + \frac{\rho^{-1} (\phi - \beta)^2}{(\theta - g^* (\phi - \beta))^2} + \frac{(\mu - 1)\mu}{(g^*)^2} \right]
\]

where \( \tilde{A} \equiv (A (1 - \phi))^\mu \). Substituting \( g^* = -\frac{\theta}{(\phi - \beta) (1 - \mu)} \) into the first bracketed expression shows this is 0. Thus

\[
\frac{\partial^2 \lambda}{\partial g} \Big|_{g=g^*} = Z\tilde{A} (g^*)^\mu \left[ \frac{\mu \rho^{-1} (\phi - \beta)}{g^* \theta - g^* (\phi - \beta)} + \frac{\rho^{-1} (\phi - \beta)^2}{(\theta - g^* (\phi - \beta))^2} + \frac{(\mu - 1)\mu}{(g^*)^2} \right]
\]

Note that \( Z\tilde{A}g^* > 0 \). Substituting in for \( g^* \) and simplifying the expression in brackets gives the condition that \( \frac{\partial^2 \lambda}{\partial g} \Big|_{g=g^*} < 0 \) if \( \mu (\rho - 1) < (1 - \mu) \) which always holds; second order conditions for a maximum at \( g = g^* \) are met.
References


