Labor market trends with balanced growth*

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Abstract

Well-known stylized facts have led to widespread use of the balanced growth concept. Recently, labor market trends including rising educational attainment and share of the labor force considered skilled have cast doubt upon balanced growth as an appropriate baseline. This paper develops a version of the neoclassical growth model that allows these labor market dynamics to occur jointly with balanced growth in output. Along the balanced growth path, skill-biased technological change drives rising skill and education levels. Relative prices of goods adjust so that growth in the value of total output is unaffected by these labor market changes. Several plausible foundations for skill-biased technological change are offered.

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1 Introduction

In his exposition on economic growth theory, Solow (1970) asks, "What are the broad facts about the growth of advanced industrial economies that a well-told model must be capable of reproducing?" In response he cites the well-known "stylized facts" noted by Kaldor (1961). The first five of these facts suggest that industrialized countries exhibit growth patterns consistent with the theoretical construct of a balanced growth path: output, employment and capital grow at a constant rate while the capital/output ratio and factor shares are constant. Although Solow had reservations about whether balanced growth is "the normal state of affairs," the neoclassical growth model is well told in part because it reproduces these facts for economies near the steady state.[1]

Innovations to the neoclassical growth model and deviations from it often try to explain a broader set of stylized facts. For example, Kaldor's sixth fact noted that productivity grows at very different rates across countries. This fact was not addressed in a satisfactory manner by the neoclassical model and in part motivated the development of endogenous growth models.[2] More recently, some economists have noted that there are several important labor market trends which a well-told model of growth should replicate. Among these are increases in average educational attainment and the share of the labor force considered skilled. In the neoclassical growth model, such trends are typically interpreted as part of the trend in technological progress or the quality of labor. However, as there are upper bounds on both education levels and the share of the population with skill, these trends cannot continue indefinitely. An eventual slowdown in the per capita growth rate might be expected as these contributions to growth inevitably diminish.[3] For example, Jones (2002) uses growth accounting techniques in a version of the neoclassical growth model to argue that the U.S. is currently in a transition phase and that the observed steady growth should be called constant growth rather than balanced growth. He claims that the stable growth suggested by the Kaldor facts is illusionary and that growth rates will begin to fall as educational achievement levels off.

In this paper we reconsider whether these labor market facts, like the facts cited by Romer

[1] Solow's view about the importance of the steady state is more accurately captured by the complete quotation from his 1970 exposition, "If it is too much to say that steady-state growth is the normal state of affairs in advanced capitalist economies, it is not too much that divergences from steady-state growth appear to be fairly small, casual, and hardly self-accentuating."

[2] Romer (1989) states additional features of the world growth experience and reviews models which have been successful in addressing these features.

[3] A similar slowing of the per capita growth rate would arise in typical endogenous growth models where schooling is directly linked to the growth rate of human capital creation.

(1989), motivate any meaningful departure from the neoclassical growth model of an economy near the steady state.[4] We find that they do not. The neoclassical model is consistent with labor market trends both on and off the balanced growth path. In our model, as these labor market dynamics fade and vanish, growth marches on. The key message is that rising education levels and skilled labor shares can arise in an economy whose growth patterns match those anticipated by the neoclassical model near the steady state. We do not need to take the existence and persistence of such trends as evidence that they matter for growth. Also, we do not need to see their imminent demise as foretelling a slowdown in growth.

We demonstrate the possibility of balanced growth with trends in the labor market by introducing skill-biased technological change into an otherwise standard neoclassical growth model with two sectors and three factors of production: capital, skilled labor and unskilled labor.[5] Skill-biased technological change drives rising skill and education levels. However, growth in the value of total output is unaffected by these labor market changes. This occurs because factor mobility and intratemporal optimization adjust the relative prices of goods to reflect their labor content. We show that this allows the value of output to be inferred from the size of the capital stock. Furthermore, preferences are such that the relative price changes in the consumption goods sector have no impact on the share of output that is saved. With the value of output and the share of output that is saved both independent of labor market changes, balanced growth can be consistent with labor market trends.

Although a simple two-good model is sufficient to produce the result, we find it useful to present the model more generally with many good types. The multi-good environment includes a structure for the introduction of new goods and this allows a more attractive interpretation regarding the course of technological change. Our multi-good structure shares some similarities with other dynamic multi-good models such as Aghion and Howitt (1992) and Grossman and Helpman (1991), but because our focus is on labor dynamics, we simplify the product introduction process and assume that it occurs exogenously.

We are able to describe the dynamics of our model economy both in transition and along the balanced growth path. Since there is some consensus that the concept of balanced growth is roughly

[4] Kongsamut, Rebelo and Xie (2001) also address balanced growth with labor trends, but instead focus on the labor allocation trend indicating a shift in labor from the agricultural sector to the service sector.

[5] There is a large literature in which labor heterogeneity coexists with growth. Notable examples include Saint-Paul (1996), Wasmer (1999) and Duranton (2001).

supported by 100 years of data, we focus on this circumstance. We show that with balanced growth in the goods market, it is possible to achieve a wide range of labor dynamics. In particular, we show that if technology evolves to favor skilled labor on average, it is possible to have balanced growth in the goods market with relative increases in skill and education.

We interpret the skill bias in technology as arising from three plausible, though not exhaustive, sources. First we consider skill-biased technological change across identical consumption goods. Next, we consider a case we call simplifying-by-doing. In this specification new goods are introduced with relatively high skilled labor shares, but over time the skilled labor share for any particular good declines while the unskilled labor share increases. These changes arise as repetition leads to more efficient production techniques whereby firms are able to substitute the low-cost unskilled labor for the higher-cost skilled input. At the product level, the simplifying-by-doing structure results in a decreased need for skilled labor. However, skilled labor's share in total output rises because the continuous introduction of new products with high initial skill requirements offsets the skill declines occurring at the product level. Finally, we consider a case we call vintage goods where skill requirements for any particular good do not change over time, but new goods are more skill intensive. This leads to an overall increase in skilled labor's share in total output.

The remainder of the paper is structured as follows. In section 2 we explain the empirical facts and the potential problem for growth theory. Section 3 describes our general model and Section 4 shows several general results implied by the model. It is in this section where we show our formulation implies that one can consider the goods market dynamics independently of the labor market dynamics. In Section 5 we focus on labor market implications under the three labor share formulations. Here it is shown that each of the formulations leads to labor market dynamics in which there are rising education levels and rising fractions of the labor force that are skilled. Finally, Section 6 summarizes and concludes the paper.

2 The data of balanced growth and labor market trends

The empirical facts documented by Kaldor over fifty years ago have withstood the test of time and continue to bear upon research in economic growth. Figures 1a and 1b show the essence of the Kaldor facts based on the most recent data. Appendix B gives details of the data. Figure 1a illustrates that, although it is variable, there is no trend in the growth rate of GDP. The figure also plots the real interest rate which we interpret as a proxy for the real return on capital. Again there is no trend. Figure 1b plots several of the so called great ratios.[6] Here we see that the share of output used to compensate labor has remained relatively steady and the ratios of capital and investment to output exhibit no trend. These facts have been interpreted by growth economists from Solow (1970) to Romer (1987, 1990) as indicating an economy growing along a balanced growth path.

More recently, it has been pointed out that other empirical facts may contradict the balanced growth paradigm. One notable trend has been in the skill level of the population. Figure 2 illustrates this type of trend. Here several measures of schooling, which we interpret as indicators of the skill level of the population, are seen to increase over time. The percent of the population with high school degrees has risen from 24.5 percent in 1940 to 84.1 percent in 2002 while the percent of the population with college degrees has risen from 4.6 to 26.7 over the same time. Figure 2 also shows that the percent of the 20-21 year old population enrolled in school has increased from 18.8 percent in 1959 to 47.8 percent in 2002.

Some economists have interpreted these trends as being incompatible with the balanced growth concept. The potential problem is simple and intuitive. Consider a common production function used in the growth literature where period t output, Y_t, is produced according to $Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$ where K_t is the capital stock, L_t is a measure of the labor input and A_t is a measure of the current state of labor augmenting technology. The parameter $\alpha \in [0, 1]$ gauges the relative importance of these factors in production. Output growth in this economy can be decomposed into $g_{Y,t} = \alpha g_{K,t} + (1-\alpha) g_{A,t} + (1-\alpha) g_{L,t}$ where $g_{Y,t}$ is the growth rate of output from period $t-1$ to period t and the other growth rates are similarly defined. If L_t is quality adjusted labor, an increase in educational attainment shows up in g_L. Otherwise it shows up in g_A. Either way, if increases in educational attainment slow, growth will slow as g_L or g_A falls.[7] The effect is even larger if g_K is related to g_A and g_L as in the Solow model and other standard models of economic growth. Since average educational attainment is clearly bounded, it seems reasonable to assume that as this contribution to growth diminishes, the overall rate of growth will fall.[8] This is at odds with the notion that the economy is growing at its balanced growth rate. In what follows we show that this

[6] We normalized the capital to output ratio by dividing the entire series by the value in 1947. This keeps the units for the three plots similar without altering the demonstration that the capital to output ratio has been fairly constant.

[7] Most growth accounting exercises that take this approach find that increases in average educational attainment account for a considerable share of economic growth in the U.S. See for example Denison (1985), Jorgenson, Gollop and Fraumeni (1987), and the recent discussion by Barro and Sala-i-Martin (2004-chapter 10).

[8] Jones (2002) conducts a more sophisticated growth accounting exercise but reaches a similar conclusion.

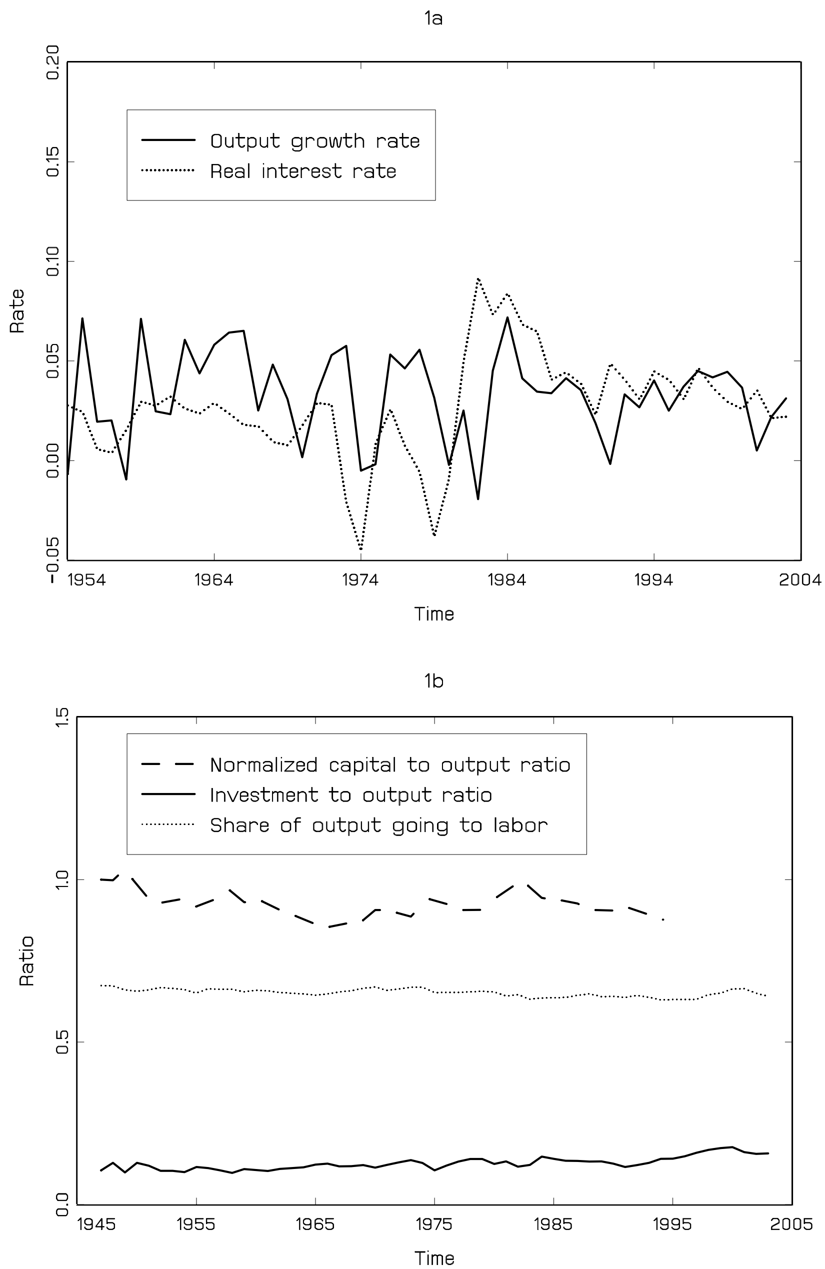

Figure 1: The stylized facts of balanced growth.

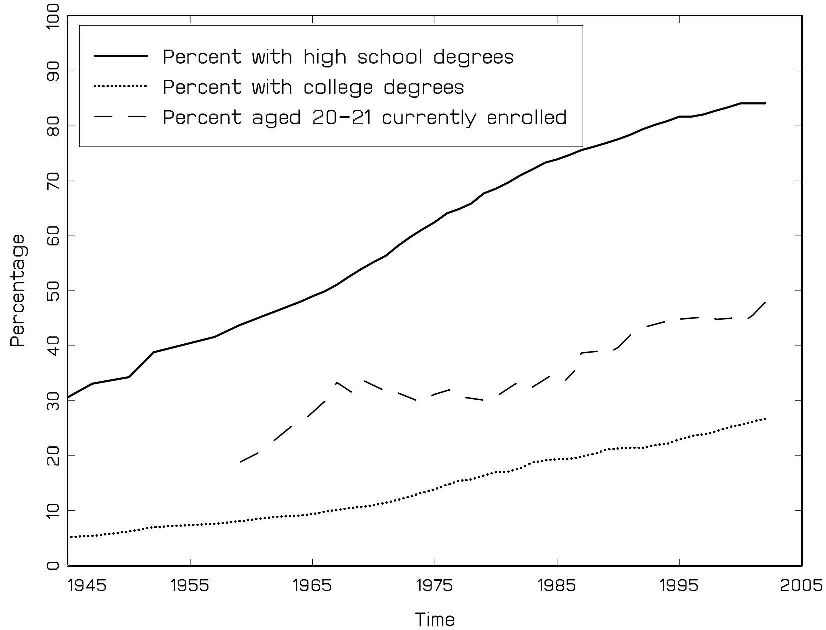

Figure 2: The labor market trends.

conclusion need not be true.

3 The model

The structure of the model is influenced by a desire to impose as little additional structure on the neoclassical growth model as necessary to match the empirical facts. Consistent with the desire to stay close to familiar ground, the model is formulated in an intensive form context with exogenous growth.[9] It is necessary to break new ground in the structure for labor and the obvious first step is to disaggregate labor into a skilled and unskilled component. Since education trends are also part of the empirical facts, an education structure is included. The major deviation from the standard neoclassical structure is to include a mechanism for the labor shares to change over time. Later, in Section 5, we show that the most palatable labor share formulations include many consumer good types. Thus from the outset, the model is described with consumer good variety.

Consumer good variety encourages a few other choices on how to structure the model. With many goods, it becomes important to be able to aggregate the different types of goods to determine a value for total output. Quite naturally total output is determined by weighing individual product

[9] An appendix describing an aggregate form of the model and its conversion into the intensive form can be obtained from the authors upon request.

outputs by their prices. With prices playing a central role in the total output calculations, a competitive equilibrium setting is the most natural. In the next few pages, we describe a two-labor-type, multi-good version of the neoclassical growth model in a competitive equilibrium setting.

3.1 The corporate sector

The corporate sector consists of two types of producers, those producing investment goods and those producing consumer goods. All goods are produced using three input types: capital, skilled labor and unskilled labor. These inputs are freely mobile between firms, thus forcing price equality for each input type across firms.

To simplify the exposition, we use the standard convention that there is only one firm that produces each type of good. This convention is justified in settings in which there are constant returns to scale for the inputs as we have here. The investment good firm uses a technology given by

$$y_t(\iota) = k_t(\iota)^{\alpha} \left[\gamma(\iota) s_t(\iota)^{\sigma} + (1-\gamma(\iota))\, u_t(\iota)^{\sigma}\right]^{\frac{1-\alpha}{\sigma}} \tag{1}$$

where ι identifies variables used in the investment producing sector. Parameter restrictions are $0 \leq \gamma(\iota) \leq 1$, $0 \leq \alpha \leq 1$ and $\sigma \leq 1$. Thus output, $y_t(\iota)$, is a Cobb-Douglas combination of capital, $k_t(\iota)$, and the labor aggregate, $\left[\gamma(\iota)s_t(\iota)^{\sigma} + (1-\gamma(\iota))\, u_t(\iota)^{\sigma}\right]^{\frac{1}{\sigma}}$. The elasticity of output with respect to the capital input is α while the elasticity of output with respect to the labor aggregate is $(1-\alpha)$. Within the labor aggregate, skilled labor, $s_t(\iota)$, and unskilled labor, $u_t(\iota)$, are combined in such a way that their elasticity of substitution is constant and governed by σ. The relative importance of each in production is governed by $\gamma(\iota)$. In the special case where $\sigma = 0$, the labor aggregate is $s_t(\iota)^{\gamma(\iota)} u_t(\iota)^{(1-\gamma(\iota))}$.

There is a growing continuum of consumer goods, each produced by a single firm. The number of goods (and firms) at time t is denoted by n_t and grows according to $n_t = n_0 e^{gt}$ where $g > 0$. We will use $\omega \epsilon [0, n_t]$ to denote a typical consumer good that is available at date t. There is a considerable literature beginning with Spence (1976) and Dixit and Stiglitz (1977) and including Romer (1987, 1990), Grossman and Helpman (1991) and Aghion and Howitt (1992) which models the foundations of product diversity and in the later cases links these foundations to entrepreneurship and growth. Using similar foundations would be distracting to the objectives here. Instead, we focus on this simple exogenous product creation process.

The production specification for a typical consumer good ω is given by

$$y_t(\omega) = k_t(\omega)^{\alpha} \left[\gamma_t(\omega) s_t(\omega)^{\sigma} + (1-\gamma_t(\omega)) u_t(\omega)^{\sigma}\right]^{\frac{1-\alpha}{\sigma}} \tag{2}$$

where $y_t(\omega)$, $k_t(\omega)$, $s_t(\omega)$ and $u_t(\omega)$ denote the output level, capital input, skilled labor input and unskilled labor input for product ω at time t. We assume $0 \leq \gamma_t(\omega) \leq 1$. Again elasticities of output with respect to capital and the labor aggregate are α and $(1-\alpha)$. In equilibrium, the share of output paid to skilled and unskilled labor will be governed by $\gamma_t(\omega)$. For example, in the special case where $\sigma = 0$, the share of output paid to skilled labor is $\gamma_t(\omega)(1-\alpha)$ and the share paid to unskilled labor is $(1-\gamma_t(\omega))(1-\alpha)$. Thus to facilitate our discussion, we refer to $\gamma_t(\omega)$ as the share parameter for good ω at time t. This formulation allows the possibility for consumer goods to have different output elasticities with respect to skilled and unskilled labor both across goods and across time. Later, in Section 5, we specify several functional formulations for share parameters and investigate their implications for labor market dynamics. For now it is sufficient to note that we allow these differences to exist.

In the above production functions, the Cobb-Douglas component holds labor income as a share of output constant at $(1-\alpha)$. As is common in the growth literature, its use here is motivated by the observed lack of trend in this share. The constant elasticity of substitution form of the labor aggregate is motivated by the empirical observation that skilled and unskilled labor appear to be more substitutable than the Cobb-Douglas specification allows.[10]

It is assumed that these firms compete in both input and product markets. The various corporate optimization problems combined with input mobility result in the following input prices:

$$r_t(\iota) = r_t(\omega') = r_t(\omega) = \alpha \frac{p_t(\omega) y_t(\omega)}{k_t(\omega)} = \alpha \frac{p_t(\iota) y_t(\iota)}{k_t(\iota)}, \tag{3}$$

$$w_t^s(\iota) = w_t^s(\omega') = w_t^s(\omega) = \frac{(1-\alpha)\gamma_t(\omega) p_t(\omega) y_t(\omega) s_t(\omega)^{\sigma-1}}{\gamma_t(\omega) s_t(\omega)^{\sigma} + (1-\gamma_t(\omega)) u_t(\omega)^{\sigma}}, \tag{4}$$

$$w_t^u(\iota) = w_t^u(\omega') = w_t^u(\omega) = \frac{(1-\alpha)(1-\gamma_t(\omega)) p_t(\omega) y_t(\omega) u_t(\omega)^{\sigma-1}}{\gamma_t(\omega) s_t(\omega)^{\sigma} + (1-\gamma_t(\omega)) u_t(\omega)^{\sigma}}, \tag{5}$$

where $p_t(\iota)$ denotes the price of an investment good, $p_t(\omega)$ denotes the price of consumer product ω, and the ω' notation denotes any other consumer good. In the following analysis we use capital as the numeraire good. Since an ex-dividend unit of capital is equivalent to a unit of the investment good, choosing capital as the numeraire requires $p_t(\iota) = 1$. We impose this condition throughout.

[10] See Katz and Murphy (1992) and Blankenau (1999).

3.2 The consumer sector

This sector consists of an infinitely lived representative consumer with lifetime utility given by

$$\int_0^\infty e^{-\rho t} \ln(c_t) dt, \tag{6}$$

where $0 < \rho$ and c_t is an index of current consumption. This index is a CES combination of all goods consumed in period t, and is given by

$$c_t \equiv \left[\frac{1}{n_t} \int_0^{n_t} x_t(\omega)^{1-\psi} d\omega \right]^{\frac{1}{1-\psi}}, \tag{7}$$

where $x_t(\omega)$ indicates the demand for good ω at date t. The parameter $\psi > 0$ gauges the intratemporal elasticity of substitution across goods and it can be shown that as ψ approaches 1, this converges to

$$c_t \equiv \exp \left[\frac{1}{n_t} \int_0^{n_t} \ln x_t(\omega) d\omega \right]. \tag{8}$$

The consumer maximizes utility subject to several types of constraints. First they face a goods constraint at each date given by

$$\begin{aligned} \dot{k}_t &= r_t(\iota)k_t(\iota) + w_t^s(\iota)s_t(\iota) + w_t^u(\iota)u_t(\iota) \\ &+ \int_0^{n_t} [r_t(\omega)k_t(\omega) + w_t^s(\omega)s_t(\omega) + w_t^u(\omega)u_t(\omega)]d\omega \\ &- \int_0^{n_t} x_t(\omega)p_t(\omega)d\omega - (\delta + g_A + g_L)\, k_t \end{aligned} \tag{9}$$

where k_t is the total capital stock at date t, $0 < \delta < 1$ is the depreciation rate of capital, g_A is the exogenous rate of technological progress and g_L is the exogenous rate of population growth.[11] After substituting in market equilibrium conditions described below, this formula exhibits the standard dynamic capital formulation. In particular, the first three terms to the right of the equality and the integral in the second line are collectively total income. The integral in the third line is total spending on consumption goods. The difference between income and consumption spending is investment which we denote by i_t. Given this, equation (9) states that the change in capital equals investment minus the amount of capital that depreciates through productive activities, i.e. $\dot{k}_t = i_t - (\delta + g_A + g_L)\, k_t$.

Throughout the paper, we use the terminology "total", as in total capital stock, to indicate the sum across all goods.[12] When the summation involves a variable which has the same value across

[11] More precisely k_t is capital 'per effective labor unit.' Throughout we drop this qualifier when referring to variables in intensive form.

[12] We decided against using the term "aggregate" since this is not appropriate for the intensive form context used here.

goods, the total will be a non-weighted sum, while for variables that vary in value across goods the summation will be price weighted. Thus, the total capital stock is given by

$$k_t = k_t(\iota) + \int_0^{n_t} k_t(\omega)\, d\omega, \tag{10}$$

while total output is given by

$$y_t = y_t(\iota) + \int_0^{n_t} y_t(\omega)\, p_t(\omega)\, d\omega. \tag{11}$$

The agent has one unit of time in each period. Since leisure provides no utility, the agent markets this time to maximize the income generated through providing skilled and unskilled labor. Skill can be defined quite broadly. The meaningful requirement is that its provision necessitates some costly refinement of the labor endowment. This cost creates an equilibrium wedge between the skilled wage and the unskilled wage. That is, a skilled wage premium arises which just compensates the cost of skill. The cost could be in terms of goods but we prefer to state it in terms of time.[13] In particular, time must be devoted to education for each unit of skill provided. The cost of education, then, is in terms of foregone wages. It is the presence of this cost, not its timing or specifics, that matters for our later analysis. Thus we opt for simplicity and model the education requirement for skill as linear and contemporaneous. In particular the total amount of skill provided currently is subject to:

$$S_t \equiv s_t(\iota) + \int_0^{n_t} s_t(\omega) d\omega = \theta e_t, \tag{12}$$

where $\theta > 0$ is the inverse cost of skill, e_t is the share of time spent in education and S_t is the share of time currently devoted to providing skilled labor. Remaining time is provided as unskilled labor giving the following time constraint:

$$u_t(\iota) + \int_0^{n_t} u_t(\omega) d\omega + s_t(\iota) + \int_0^{n_t} s_t(\omega) d\omega + e_t = 1. \tag{13}$$

The first two items are collectively the total unskilled labor which we denote U_t.

With this model description, a competitive equilibrium is defined as follows:

Definition: A competitive equilibrium is a set of infinite price sequences $\{r_t(\iota), w_t^s(\iota), w_t^u(\iota), [p_t(\omega), r_t(\omega), w_t^s(\omega), w_t^u(\omega), 0 \leq \omega \leq n_t] : 0 \leq t\}$ and infinite allocation sequences

[13] Modelling education as a time cost is common. See Lucas (1988) for example. Milesi-Ferretti and Roubini (1998) show that whether education is a time cost or a goods cost can matter in analyzing tax policy. As we abstract from issues of taxation, the nature of the cost is less important.

$\{k_t, y_t(\iota), i_t, k_t(\iota), s_t(\iota), u_t(\iota), [y_t(\omega), x_t(\omega), k_t(\omega), s_t(\omega), u_t(\omega), 0 \leq \omega \leq n_t], e_t : 0 \leq t\}$ with k_0 given such that:

1. Given prices, firms maximize profits subject to production constraints (1) and (2). With factor mobility this yields (3), (4), and (5).

2. Given prices, consumers maximize utility (6) subject to resource constraints (9), (12) and (13).

3. Markets clear:

   (a) Capital goods produced equals investment: $y_t(\iota) = i_t$.

   (b) Consumption goods produced equals consumption good demand: $y_t(\omega) = x_t(\omega)$ for $0 \leq \omega \leq n_t$.

   (c) Capital input demand equals capital input supplied: (10).

   (d) Labor input demand equals labor input supplied: (12) and (13).[14]

4 Implications of the general model

In this section we focus on a general proposition showing the equilibrium allocations of capital and time across investment and consumption goods. Although the proposition provides information on a host of equilibrium behavior, one implication is particularly important. The proposition shows that the dynamics of the capital stock and its allocation across sectors can be tracked without knowledge of the time allocations in the goods sector. An important consequence is that the dynamic behavior of total output can be tracked without knowledge of time allocations. Stated somewhat differently, the proposition implies that one can investigate time allocations without accounting for feedback into total capital and total output. In Section 5, we take this approach by developing several labor market structures without considering the impact of the different specifications on total output.

To describe the equilibrium dynamics, it proves convenient to introduce ν_t as a measure of the share of the capital stock used in the production of investment goods.

Proposition 1. Household Allocations.

(a) Capital allocations: There exits a ν_t such that capital allocations are

$$k_t(\iota) = \nu_t k_t \tag{14}$$

[14] Equations (12) and (13) appear as both constraints to the consumer and labor market clearance conditions because we have assumed there to be a single representative agent.

and

$$k_t(\omega) = (1 - \nu_t)\, k_t z_t(\omega)^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} Z_t^{-1} \tag{15}$$

and such that the dynamics of ν_t and k_t are governed by

$$\frac{\dot{\overline{1 - \nu_t}}}{1 - \nu_t} = \alpha k_t^{\alpha-1} z(\iota)^{(1-\sigma)(1-\alpha)} (1 - \nu_t) - (1 - \alpha)(\delta + g_A + g_L) - \rho, \tag{16}$$

$$\frac{\dot{k}_t}{k_t} = z(\iota)^{(1-\sigma)(1-\alpha)} \nu_t k_t^{\alpha-1} - (\delta + g_A + g_L), \tag{17}$$

where $z(\iota) \equiv \left[\gamma(\iota)^{\frac{1}{1-\sigma}} \left(\frac{\theta}{1+\theta}\right)^{\frac{\sigma}{1-\sigma}} + (1 - \gamma(\iota))^{\frac{1}{1-\sigma}}\right]^{\frac{1}{\sigma}}$,
$z_t(\omega) \equiv \left[\gamma_t(\omega)^{\frac{1}{1-\sigma}} \left(\frac{\theta}{1+\theta}\right)^{\frac{\sigma}{1-\sigma}} + (1 - \gamma_t(\omega))^{\frac{1}{1-\sigma}}\right]^{\frac{1}{\sigma}}$,
$Z_t \equiv \int_0^{n_t} [z_t(\omega)]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} d\omega$,
and where equations (16) and (17) describe a globally stable system converging to a path with $\frac{\dot{\overline{1-\nu_t}}}{1-\nu_t} = \frac{\dot{k}_t}{k_t} = 0$.

(b) Labor allocations: Skilled and unskilled labor inputs at the firm level are given by

$$u_t(\omega) = [1 - \gamma_t(\omega)]^{\frac{1}{1-\sigma}} \frac{(1 - \nu_t)\, z_t(\omega)^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}}{Z_t} \tag{18}$$

$$s_t(\omega) = \left[\gamma_t(\omega) \frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}} \frac{(1 - \nu_t)\, z_t(\omega)^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}}{Z_t} \tag{19}$$

$$u_t(\iota) = \nu_t \frac{(1 - \gamma(\iota))^{\frac{1}{1-\sigma}}}{z(\iota)^{\sigma}} \tag{20}$$

$$s_t(\iota) = \nu_t \frac{\left[\gamma(\iota) \frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}}}{z(\iota)^{\sigma}} \tag{21}$$

Proof: See the Appendix A.

The proposition provides results on the equilibrium behavior of capital and its allocation in part (a). The allocation of capital within the goods sector depends upon the exogenous state variables Z_t and $\gamma_t(\omega)$ as seen in equation (15). However, since $\int_0^{n_t} k_t(\omega) d\omega = (1 - \nu_t)\, k_t$, these parameters do not influence capital's allocation between the capital good and consumption good sectors. Part (b) and equation (13) describe the allocation of time across skilled labor, unskilled labor and education. To find time allocations in the goods sector requires knowledge of Z_t and $\gamma_t(\omega)$. Since these are absent from equations (16) and (17), the dynamics of both capital and its allocation can be tracked without reference to time allocations in the goods sector. The only information required from the labor market is $z(\iota)$ which is constant and parametric. The separation does not work in the other direction. Equations (18) and (19) show that knowledge of ν_t is required for time allocations.

The proof to Proposition 1 gives the formal details for the separation result. Here we provide a more intuitive explanation as to why this separation arises. There are two key insights for understanding the separation. First, intratemporal optimization and factor mobility cause investment to depend only on ν_t and not directly on current values of Z_t and $\gamma_t(\omega)$. These same features make output and the total value of consumption independent of Z_t and $\gamma_t(\omega)$ or equivalently independent of time allocations in the goods sector. This is an intratemporal separation result since none of these aggregates depend directly on current allocations of time in the goods sector. Second, logarithmic preferences across the consumption aggregate assures that ν_t is independent of changes in Z_t and $\gamma_t(\omega)$ intertemporally. Thus investment, $\frac{\dot{k}_t}{k_t}$, output and the total value of consumption are separated from Z_t and $\gamma_t(\omega)$ intertemporally. In the next two subsections, we provide additional detail on the intratemporal and intertemporal separation. We then follow this with a discussion of the dynamics for the model and implications for an alternative formulation of the education production function.

4.1 Intratemporal separation

To demonstrate and build intuition for the intratemporal separation we will first show that the percentage of time allocated to capital good production is given by ν_t. This in combination with (14) will imply that upon entering a time period, all that is needed to determine capital good production are the current k_t and ν_t. Parameters or input allocations from the consumption good market will be unimportant. To this end, arrange equations (3)and (5) to get

$$\frac{k_t(\iota)}{u_t(\iota)} = \frac{k_t(\omega)}{u_t(\omega)} \frac{1 + \frac{\gamma(\iota)}{(1-\gamma(\iota))}\left(\frac{s_t(\iota)}{u_t(\iota)}\right)^{\sigma}}{1 + \frac{\gamma_t(\omega)}{(1-\gamma_t(\omega))}\left(\frac{s_t(\omega)}{u_t(\omega)}\right)^{\sigma}}. \tag{22}$$

Next, define $l_t(\omega) \equiv u_t(\omega) + s_t(\omega) + \frac{s_t(\omega)}{\theta}$. This is the total amount of time allocated to good ω where the final term acknowledges that in order to provide $s_t(\omega)$ units of skill, $\frac{s_t(\omega)}{\theta}$ units of time must first be spent in education. From this

$$u_t(\omega) \equiv l_t(\omega)\left(1 + \frac{s_t(\omega)}{u_t(\omega)}\left(1 + \frac{1}{\theta}\right)\right)^{-1}; \quad u_t(\iota) \equiv l_t(\iota)\left(1 + \frac{s_t(\iota)}{u_t(\iota)}\left(1 + \frac{1}{\theta}\right)\right)^{-1}. \tag{23}$$

Devoting a unit of time to skill is worthwhile only if the skilled wage compensates for the value of $\left(1+\frac{1}{\theta}\right)$ units of time that could have been spent providing unskilled labor. Thus the optimal time allocation requires

$$w_t^s(\omega) = \left(1+\frac{1}{\theta}\right) w_t^u(\omega). \tag{24}$$

Equations (4), (5) and (24) give

$$\frac{\gamma_t(\omega)}{(1-\gamma_t(\omega))}\left(\frac{s_t(\omega)}{u_t(\omega)}\right)^{\sigma} = \frac{s_t(\omega)}{u_t(\omega)}\left(1+\frac{1}{\theta}\right); \quad \frac{\gamma(\iota)}{(1-\gamma(\iota))}\left(\frac{s_t(\iota)}{u_t(\iota)}\right)^{\sigma} = \frac{s_t(\iota)}{u_t(\iota)}\left(1+\frac{1}{\theta}\right) . \tag{25}$$

Putting equations (23) and (25) into (22) gives $\frac{k_t(\iota)}{l_t(\iota)} = \frac{k_t(\omega)}{l_t(\omega)}$. Thus intratemporal optimization and factor mobility assure that for all goods, the ratio of capital to time (inclusive of the requisite education time) is equal. An implication is that $k_t(\iota)\int_0^{n_t} l_t(\omega)\,d\omega = l_t(\iota)\int_0^{n_t} k_t(\omega)d\omega$. Since the time constraint requires $\int_0^{n_t} l_t(\omega)\,d\omega + l_t(\iota) = 1$, this simplifies to $l_t(\iota) = \nu_t$.

This expression yields the critical insight into the intratemporal separation. It tells us that once we know ν_t, we know the percentage of both the capital resources and the time resources that are allocated to capital good production irrespective of the consumption good parameters or input allocations. Furthermore (25) and $l_t(\iota) \equiv u_t(\iota) + s_t(\iota) + \frac{s_t(\iota)}{\theta}$ tell us how this time is allocated within the capital good sector, so knowing k_t and ν_t becomes sufficient to determine the capital good output level.

An alternative demonstration that capital good production is independent of consumption good parameters can be found by substituting (20), (21) and (14) into (2) to get

$$i_t = z(\iota)^{(1-\sigma)(1-\alpha)}\nu_t k_t^{\alpha}.$$

Other parts of the goods market are also independent of the consumption good parameters. Equation (3) assures $p_t(\omega)y_t(\omega) = k_t(\omega)\frac{i_t}{k_t(\iota)}$. Integrating both sides across consumption goods gives

$$c_t = (1-\nu_t)\, z(\iota)^{(1-\sigma)(1-\alpha)} k_t^{\alpha}$$

so we can also find the value of the consumption aggregate from knowledge of ν_t and k_t. More intuitively, since ν_t is the share of both time and capital devoted to producing the investment good, $(1-\nu_t)$ is the share of resources devoted to producing the consumption goods. The relative value of total consumption ultimately reflects the relative value of these inputs. Thus $\frac{c_t}{i_t} = \frac{\nu_t}{1-\nu_t}$ and with i_t known, c_t is known. Output, the sum of consumption and investment, is simply

$$y_t = z(\iota)^{(1-\sigma)(1-\alpha)} k_t^{\alpha}.$$

The value of output reflects the total value of the inputs in terms of the investment good and thus is independent of its allocation.

To summarize, factor mobility assures that the share of time devoted to producing the investment good is equal to the share of the capital stock put to this use. Since the relative price of skilled and unskilled labor is fixed, the allocation of these ν_t units of time is known. Knowledge of ν_t and k_t identifies the quantities of all resources used in investment and thus identifies i_t. The remaining resources are used in producing the various consumption goods and the aggregate value of these consumption goods reflects the value of these resources. Given this, we can find the value of aggregate consumption without knowledge of specific input allocations in the goods sector.

4.2 Intertemporal separation

While current investment and consumption depend only on ν_t and k_t, the separation is not complete unless changes in ν_t are unrelated to changes in Z_t. To show intuitively why this holds, we first provide new interpretations of ν_t and Z_t. Since $\nu_t = \frac{i_t}{y_t}$, it can be thought of as the savings rate. In Appendix A we show that

$$p_t(\omega) = \frac{z(\iota)^{(1-\sigma)(1-\alpha)}}{z_t(\omega)^{(1-\sigma)(1-\alpha)}}. \tag{26}$$

Using this and the definition of Z_t from the proposition gives

$$Z_t = z(\iota) \int_0^{n_t} p_t(\omega)^{\frac{-(1-\psi)}{\psi}}$$

so that Z_t is a composite function of prices. What is needed, then, is for the dynamics of the savings rate to be unrelated to the dynamics of the consumption goods prices. But the independence of the savings rate from consumption goods prices is a well-know characteristic of intertemporally unit elastic (logarithmic) preferences.[15] Note that such a restriction on the intratemporal elasticity substitution as gauged by ψ is not required.

4.3 Dynamics

There are two sources of dynamics in this model. Transitional dynamics enter through changes in ν_t and k_t while technology dynamics enter through changes in Z_t and $\gamma_t(\omega)$. So long as ν_t and k_t change through time, transitional dynamics are part of the reason for changes in the allocation of resources as seen in equations (14) through (21). However the final line of part (a) in the

[15] In the unpublished technical appendix we develop the more general case where $U = \int_0^\infty e^{-\rho t} \frac{c_t^{1-\lambda}-1}{1-\lambda} dt$ and show that balanced growth occurs only with $\lambda = 1$ (the log case) or $\dot{Z}_t = 0$.

proposition states that the economy converges to a path where $\frac{\dot{\overline{1-\nu_t}}}{1-\nu_t} = \frac{\dot{k}_t}{k_t} = 0$ so that transitional dynamics eventually cease. Since the savings rate, the value of output per effective unit of labor, the value of consumption, and $\frac{k_t(\iota)}{l_t(\iota)} = \frac{k_t(\omega)}{l_t(\omega)}$ are all constant along this path, we refer to this as the balanced growth path.[16] Furthermore, from equation (3) the real interest rate is constant.[17] Thus the concept of generalized balanced growth as introduced by Kongsamut, Rebelo and Xie (2001) is satisfied when there are no transitional dynamics.

Transitional dynamics are discussed more in the proof to Proposition 1. However, since our primary objective is to demonstrate the possibility of labor market trends which are not rooted in transitional dynamics, the remainder of the paper focuses on situations in which there is balanced growth. The main point is that a rich set of technology dynamics continues when transitional dynamics die out and these dynamics can be consistent with observed labor market trends.

Consider first, equations (18) and (19). These show that even with ν_t constant, the amount of skilled and unskilled labor employed in the production of each consumption good can change. Thus while total time spent in the goods sector is fixed in balanced growth, its allocation across the various goods responds to technological change. This occurs through the term Z_t even for goods whose own technology parameters are fixed. For a particular good the mix of time across skilled, unskilled labor and education can be found using (18) and (19). Though relative wages are fixed, this allocation changes with $\gamma_t(\omega)$. Similarly, equation (15) shows that even with the share of capital used in consumption goods constant, its allocation across consumption goods is influenced by the technology parameters. Given these expressions for inputs and given equation (2) the output of good ω is equal to

$$x_t(\omega) = \frac{z_t(\omega)^{\frac{(1-\alpha)(1-\sigma)}{\psi}}}{Z_t}(1-\nu_t)\,k_t^{\alpha}. \tag{27}$$

Thus the output of particular goods can follow rich dynamics even in balanced growth; i.e. with ν and k constant. Furthermore, equation (26) shows that any good experiencing technological change will experience a change in its equilibrium price.

The technology dynamics of greatest importance for this paper are those of the total supply of skilled labor and the share of the population involved in education, S_t and e_t. Our separation result

[16] We will use the term balanced growth here even though the intensive form has no growth. This is consistent with the common terminology and interpretation of the Solow model.

[17] Here we use the definition of real interest rate introduced by Kongsamut, Rebelo and Xie (2001) of real interest rate $= r_t(\iota) - \delta$.

allows us to have any sort of labor market dynamics in the goods sector. Clearly then, with the current level of generality, it is possible to have S_t and e_t both increase along the balanced growth path. The key implication is that a fairly standard exogenous growth model can match features of the observed labor market trends without relying on transitional dynamics. Hence such trends alone are an insufficient argument against the balanced growth paradigm.

4.4 Non-linear skill production and wage inequality

Before moving on to study a variety of special cases, it is useful to step back and explore the implications of relaxing the linear education formulation. Consider generalizing equation (12) so that $S_t = \theta e_t^{\eta}$. Here $\eta > 0$ gauges the marginal product of education measured in units of skill. Optimal education again requires that the skilled wage just compensates for the value of time that could have been supplied as unskilled labor. However, $\frac{1}{\theta \eta e_t^{\eta-1}}$ units of time are now required to provide additional skill at the margin. Thus equation (24) becomes

$$\frac{w_t^s(\omega)}{w_t^u(\omega)} = \left(1 + \frac{1}{\theta \eta e_t^{\eta-1}}\right) \tag{28}$$

and $l_t(\iota) \equiv u_t(\iota) + s_t(\iota) + \frac{1}{\theta \eta e_t^{\eta-1}} s_t(\iota)$. Otherwise the argument from equations (22) through (25) is identical, so $l_t(\iota) = \nu_t$.

This generalization, then, is not problematic for identifying the amount of time spent on the investment good. However, note that when $\theta \eta e_t^{\eta-1}$ replaces θ in equation (25), the allocation of the ν_t units of time depends on e_t. Thus $s_t(\iota)$ and $u_t(\iota)$ are not constant when e_t is trended. This in turn means that investment is not fixed and balanced growth is not possible without imposing further restrictions.

The earlier implicit assumption that $\eta = 1$ is one way to assure that allocation of the ν_t units of time is fixed. A straightforward alternative is to set $\gamma(\iota) = 0$ so that the investment good is produced using only capital and unskilled labor. In this case, $s_t(\iota) = 0$, $u_t(\iota) = \nu_t$ and equations (16) and (17) describe an economy converging to a balanced growth path independent of input allocations in the goods sector and balanced growth can arise with labor trends as observed in the data.

One advantage of this specification is that it allows for dynamics in wage inequality. Using equation (28), if $\eta < 1$, periods of increasing educational attainment yield a rising wage ratio. With diminishing returns to education, it becomes increasingly difficult to produce additional skill and the equilibrium wedge between the skilled and unskilled wages rises to compensate the rising time cost of producing skill. Because rising wage inequality across education levels is a prominent recent feature of the U.S. labor market experience (Katz and Murphy (1992)), this specification does have some merit.

5 Illustrations of balanced growth with labor trends

Having shown that upward trends in S_t and e_t can be consistent with balanced growth in a relatively general setting, we now turn our attention to a case where the dynamics of S_t and e_t are more transparent. In particular, we consider the special case where $\sigma = 0$ and $\psi = 1$, making the labor aggregate in production a Cobb-Douglas combination of skilled and unskilled labor and the period utility function logarithmic in each good. This allows transparent results since we are able to express the dynamics of labor use by tracking only the average skill share parameter given by $\widetilde{\gamma}_t \equiv \frac{1}{n_t} \int_0^{n_t} \gamma_t(\omega)\, d\omega$. In this spirit we state the following corollary to Proposition 1.

Corollary 1. *Let* $\sigma = 0$ and $\psi = 1$. *Suppose* $\widetilde{\gamma}_t$ *increases monotonically to some constant and that the economy is in balanced growth,* $\frac{\dot{(1-\nu_t)}}{(1-\nu_t)} = \frac{\dot{k}_t}{k_t} = 0$, *then,*
(a) the share of time devoted to education rises monotonically and converges to some $\overline{e} \in (0,1)$,
(b) the share of time devoted to skilled labor rises monotonically and converges to some $\overline{S} \in (0,1)$,
(c) the share of time devoted to unskilled labor falls monotonically and converges to some $\overline{U} \in (0,1)$.

An advantage of such a simple result is that it allows us to look more deeply into the nature of the technological change behind the simultaneity of balanced growth and labor market trends.[18] The only requirement is that on average the skill share parameter increases monotonically and converges to some upper bound.

The behavior of the average skill share parameter clearly is dictated by the dynamics of the underlying good-specific skill share parameters. The question arises whether in naturally occurring economies it is reasonable to expect that these underlying parameters might follow dynamics that aggregate to a rising and converging average skill share. In the remainder of this section we describe three specific labor share functional forms which share this feature. These forms represent possibilities, but one can imagine others, including hybrids of these. Our purpose is to demonstrate

[18] Corollary 1 continues to hold with a non-linear education cost function so long as $\gamma(\iota) = 0$. The proof is in the unpublished appendix.

that labor market trends can arise when technology changes in ways that are both intuitively appealing and consistent with observations. In what follows we describe each form functionally, demonstrate that it is consistent with rising education and skill levels and describe some additional properties.

The first form, which we call "technological change in the consumption goods sector," is the simplest as it reduces the economy to a capital good and a single consumption good. While the form has some attractive properties, it has several intuitive limitations. First, balanced growth in output may be due to a rising relative price of the consumption good and a simultaneous decrease in its quantity. In addition, this form implies that over time the consumption good sector adopts production processes which are relatively more intensive in their use of costly skilled labor.

These intuitive limitations of the first labor form are not present in the other formulations where the product space expands. This is the primary reason for providing the multi-good environment. We refer to these other labor formulations as "simplifying-by-doing" and "vintage goods." In each case, newer products have higher skilled labor shares than products that were previously developed. This assumption is consistent with both anecdotal data and empirical investigations.[19]

5.1 Technological change in the consumption good sector

Our simplest and most transparent formulation sets $\gamma_t(\omega) = \widetilde{\gamma}_t$ for all goods and assumes $\dot{\widetilde{\gamma}}_t > 0$. Under this formulation, all the consumption goods have identical production technologies and identical utility values, thus effectively rendering them identical. The production technologies exhibit skill-biased technological change due to the rising output elasticities with respect to skilled labor. This bias implies that skilled labor's share of input payments rises for consumption goods relative to investment goods. There is evidence to support that technological change has not been symmetric across consumption and investment goods. Gordon (1990) constructs a quality adjusted price series for capital equipment and finds that the price of equipment capital relative to consumption has declined since 1963.[20] Greenwood, Hercowitz and Krusell (1997) and Krusell et. al. (2000) interpret the price decline as evidence of technological change in the production of equipment capital. Similarly, we can interpret this as evidence of technological change in the

[19] Adler and Clark (1991) find evidence that after a new plant is opened, there is a temporary increase in the number of engineers employed in order to get the plant "up to speed." Bartel and Lichtenberg (1987) argue their evidence shows that educated workers have a comparative advantage in implementing new technologies.

[20] We do not model equipment capital as distinct from structures capital. However, since equipment capital comprises some share of total investment, we take this as evidence for changes in the relative technologies.

production of the consumption good. Either interpretation yields a change in relative prices.

Corollary 2a summarizes the labor market dynamics and consumption good price results for this formulation. We drop the time subscript on ν to indicate its balanced growth level.

Corollary 2a. *Let the goods market be in balanced growth and let* $\gamma_t(\omega) = \widetilde{\gamma}_t$, $\forall\omega$, and $\dot{\widetilde{\gamma}}_t > 0$. *Then,*
(a) $\dot{U}_t = -(1-\nu)\dot{\widetilde{\gamma}}_t$, $\dot{S}_t = \frac{1}{1+\theta}(1-\nu)\dot{\widetilde{\gamma}}_t$ *and* $\dot{e}_t = \frac{\theta}{1+\theta}(1-\nu)\dot{\widetilde{\gamma}}_t$ and
(b) $\dot{p}_t(\omega) > 0$ *if* $\widetilde{\gamma}_t < \frac{\theta+1}{2\theta+1}$.

Result (a) of Corollary 2a shows that this simple specification can generate the type of labor market behavior observed historically in the U.S. Result (b) shows that for some values of $\widetilde{\gamma}_t$, the relative price of the consumption good will rise, matching the empirical evidence. While the results are appealing on these grounds, several issues indicate this may not be the whole story. Most importantly, a rising price of the consumption good with its total value constant implies a decrease in the number of units of the consumption goods per effective unit of labor. While this does not imply a decrease in the absolute number of consumption goods, we nonetheless consider this at odds with the finding of increased real consumption from national income accounts.[21] Of course with $\widetilde{\gamma}_t > \frac{\theta+1}{2\theta+1}$ we could have an increasing number of consumption goods with a decrease in the relative price but this is in conflict with the Gordon data.

Another weakness of this specification is that it requires that the skill share of the consumption good rises through time as experience in its production grows. Though we do not model the creation of technological advances, one would expect that most changes in production technology for an existing good would be motived by lowering costs. That is, in a richer model we would expect to see firms search for ways to substitute low cost unskilled labor for higher cost skilled labor rather than vice versa. Empirical evidence supports this notion. For example, Adler and Clark (1991) show that upon opening a new plant, there is a temporary increase in the use of engineers to get the plant up to speed. Through time this skill requirement falls. Thus the skill requirements should be expected to fall as experience is gained in its production. These weaknesses of the simplest approach motivate the next two formulations.

[21] Recall $y_t(\omega)$ is output in intensive form. The growth rate of the total output of good ω is $\frac{\dot{y}_t(\omega)}{y_t(\omega)} + g_L + g_A$ which can be positive with $\frac{\dot{y}_t(\omega)}{y_t(\omega)} < 0$.

5.2 Simplifying-by-doing

Our simplifying-by-doing formulation assumes that for any given product, as the product ages, its skilled labor share declines. We interpret this as capturing the possibility that firms learn more cost effective production techniques for producing a product over time and these cost effective techniques make greater use of the low cost unskilled labor. Under this formulation, the skill intensity of a good depends on how long it has been in production. Our formulation focuses only on developing lower cost production techniques for each good and does not consider potential changes that could arise at the consumer good frontier. This later possibility is explored in our vintage good formulation below.

Since the skill share of any particular good is declining, and the data show that the overall skill share is increasing, our formulation assumes that new products continuously enter the market and these products enter with relatively high skilled labor shares. If the reduced skill shares from older goods can be offset by the greater needs in the newer goods, then the overall skilled labor share for the economy can increase. A convenient specification of $\gamma_t(\omega)$ with these properties is

$$\gamma_t(\omega) = \underline{\gamma} + \left(\overline{\gamma} - \underline{\gamma}\right) \frac{\ln(\omega+1)}{\ln(n_t+1)}, \tag{29}$$

where $0 \leq \underline{\gamma} < \overline{\gamma} \leq 1$. To see the implications of this formulation, recall that n_t measures the product space at time t. When good ω is the frontier good, $n_t = \omega$, and its skilled labor share is given by $\gamma_f(\omega) = \overline{\gamma}$. Since n_t grows through time $\dot{\gamma}_t(\omega) < 0$. This means that each good begins with a skill share equal to $\overline{\gamma}$ which then falls through time. The fact that each good begins with a skill intensity of $\overline{\gamma}$ indicates that we do not need a rising skill share at the frontier to produce the desired result. Next note that, for any finite ω, $\lim_{t\longrightarrow\infty} \gamma_t(\omega) = \underline{\gamma}$ so that the skill share for any good converges monotonically to $\underline{\gamma}$. Finally note that $\frac{\partial \gamma_t(\omega)}{\partial \omega} > 0$, so that at any point in time, newer goods are more skill intensive.

Corollary 2b summarizes the labor market dynamics under this formulation for the labor elasticities.

Corollary 2b. *Let the goods market be in balanced growth and $\gamma_t(\omega)$ be given by equation (29). Then,*

(a) $\dot{\widetilde{\gamma}}_t > 0$ and $\lim_{t\longrightarrow\infty} \widetilde{\gamma}_t = \overline{\gamma}$ and

(b) $\dot{U}_t = -(1-\nu)\dot{\widetilde{\gamma}}_t$, $\dot{S}_t = \frac{\theta}{1+\theta}(1-\nu)\dot{\widetilde{\gamma}}_t$ *and* $\dot{e}_t = \frac{\theta}{1+\theta}(1-\nu)\dot{\widetilde{\gamma}}_t$.

Result (a) shows that the trade-off described above is working. In particular, the higher skill share required for new goods is enough to offset the decline in skill needs from each good, thus resulting in a total rise in skill shares. Result (b) shows that this implies the type of labor market trends experienced in the U.S. over time with rising skilled labor input percentages, rising education levels and falling unskilled labor input percentages.

5.3 Vintage goods

The third example is similar to the simplifying-by-doing structure in that an expanding consumer product space comes into play. But it differs in terms of how skilled labor shares change among the goods. In particular, learning does not change the technology of any particular good. Here it is assumed that $\gamma_t(\omega) = \gamma_{t'}(\omega)$, $\forall\ t', t$. What drives the results is that newer goods are more skill intensive. This would arise if new goods require more skill to invent or produce. For example, if new goods (or services) disproportionately arise in the medical, technology or professional services fields, the average new good will have a higher skill share than earlier vintages of goods. Similarly, if new goods require more skill for initial development, we can amortize these development costs over the life of the good and consider them a cost of production. This would arise, for example, if new goods are more complex or are developed using more complex techniques. Such a change would cause the skill share of new goods to be higher and is consistent with Romer's finding of a significant increases in the percentage of the labor force represented by engineers (Romer, 2000).

In this specification, each good is produced using a technology that is specific to its vintage. The concept of a vintage technology dates at least to Arrow (1962). In his specification, many vintages of technology are in use at any time. Technological advancement comes from learning-by-doing and is embodied in capital. Newer vintages require less labor input per unit of output. Our specification is related in that many vintages of technologies are simultaneously employed and once adopted the parameters of a production function are constant through time. However, our specification differs in fundamental ways. The first differences are simplifications; technological change is a function of time rather than investment and technological change is not embodied in capital. The remaining difference is motivated by our interest in labor market trends. Rather than implying a decrease in the labor requirement, we presume technology evolves exogenously to favor skilled labor. As we argue above, the natural state of affairs is that technology would change to favor the low cost input. However, in the vintage goods specification, skill-biased technological change is a consequence of the properties of new goods.

In this specification, technologies differ across firms because $\gamma_f(\omega) > \gamma_f(\omega'), \forall\, \omega > \omega'$; that is the skill share of the frontier good grows. A convenient specification for the vintage goods case is

$$\gamma_t(\omega) = \underline{\gamma} + \left(\overline{\gamma} - \underline{\gamma}\right)\left(\frac{(\phi+1)\,\omega^{\phi} + \omega^{2\phi}}{\left(1+\omega^{\phi}\right)^2}\right), \tag{30}$$

where $0 < \phi \leq 1$. Notice that this formulation does not have an n_t term and only has ω terms. This means that a product's skill intensity is not a function of time, but instead is only a function of its type. Also note that $\frac{\partial \gamma_t(\omega)}{\partial \omega} > 0$ for $0 < \phi \leq 1$, so that at any point in time, newer goods are more skill intensive.

Corollary 2c summarizes the labor market dynamics under this formulation for the labor elasticities.

Corollary 2c. *Let the goods market be in balanced growth and $\gamma_t(\omega)$ be given by equation (30). Then,*

(a) $\dot{\widetilde{\gamma}}_t > 0$ and $\lim_{t\longrightarrow\infty} \widetilde{\gamma}_t = \overline{\gamma}$

(b) $\dot{U}_t = -(1-\nu)\dot{\widetilde{\gamma}}_t$, $\dot{S}_t = \frac{\theta}{1+\theta}(1-\nu)\dot{\widetilde{\gamma}}_t$ *and* $\dot{e}_t = \frac{\theta}{1+\theta}(1-\nu)\dot{\widetilde{\gamma}}_t$.

Result (a) summarizes the expected result that as the product space expands and the total skilled labor intensity also grows. Result (b) shows that again the desired labor market trends arise.

Figure 3 shows $\widetilde{\gamma}_t$, S_t and U_t for the two latter examples. The labor market dynamics in each case are drawn with output along its balanced growth path. From these pictures, it is clear that labor supply trends can persist for a considerable period of time while output growth is balanced.[22]

6 Conclusion

This paper offers an explanation for the simultaneous observation of balanced growth in total output and trends in the allocation of time between providing skilled labor, unskilled labor and education. These observations present a puzzle for the standard neoclassical growth model in which balanced growth in the goods and capital markets require balanced behavior in the labor market. Our explanation is provided by augmenting the neoclassical growth model to include a structure for skill-biased technological change and for new goods to enter production. These changes imply that the labor market dynamics have no implication for the dynamics of total output

[22] The parameter ϕ governs the speed at which $\tilde{\gamma}$ reaches its upper bound. By decreasing ϕ, convergence to this upper bound can be made arbitrarily slow.

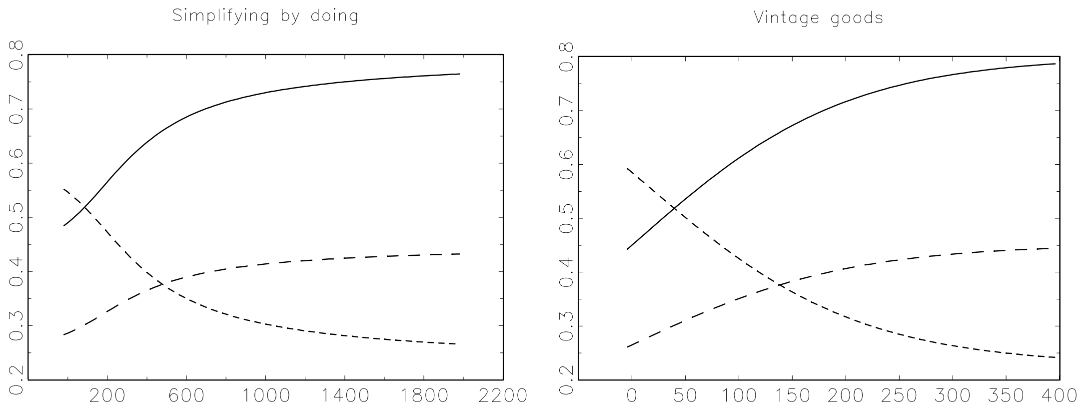

Figure 3: Labor supply dynamics along the balanced growth path. In each panel, the solid line is $\widetilde{\gamma}_t$, the long dashed line is S_t and the short dashed line is U_t. Parameter values are $\overline{\gamma} = .8$, $\underline{\gamma} = .1$, $\theta = .1$. In the second panel, $\phi = 1$.

and capital. Under several plausible labor share formulations, we show that the model is consistent with balanced growth in the goods and capital markets and labor markets in which education levels and percentages of the labor force that are skilled rise.

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A Appendix A: Proofs

A.1 Proof of Proposition 1

Write the Hamiltonian for the consumer's problem as

$$\begin{aligned}
H = {} & e^{-\rho t} \ln \left[\frac{1}{n_t} \int_0^{n_t} x_t(\omega)^{1-\psi} d\omega \right]^{\frac{1}{1-\psi}} \\
& + \mu_{1t} (r_t(\iota) k_t(\iota) + w_t^s(\iota) s_t(\iota) + w_t^u(\iota) u_t(\iota) \\
& \qquad + \int_0^{n_t} [r_t(\omega) k_t(\omega) + w_t^s(\omega) s_t(\omega) + w_t^u(\omega) u_t(\omega)] d\omega \\
& \qquad - \int_0^{n_t} p_t(\omega) x_t(\omega) d\omega - (\delta + g_A + g_L) k_t \Big) \\
& + \mu_{2t} \left(k_t - k_t(\iota) - \int_0^{n_t} k_t(\omega) d\omega \right) \\
& + \mu_{3t} \left(1 - u_t(\iota) - \int_0^{n_t} u_t(\omega) d\omega - s_t(\iota) - \int_0^{n_t} s_t(\omega) d\omega - e_t \right) \\
& + \mu_{4t} \left(s_t(\iota) + \int_0^{n_t} s_t(\omega) d\omega - \theta e_t \right),
\end{aligned}$$

and the transversality condition requires

$$\lim_{t \to \infty} \{\mu_{1t} k_t\} = 0.$$

Combining the first order conditions with (3), (4), (5) and the goods market clearance conditions, $y_t(\iota) = i_t$, $y_t(\omega) = x_t(\omega)$ for $0 \le \omega \le n_t$, gives the following system of equations for the competitive equilibrium.[23]

$$H_{x_t(\omega)} : \frac{e^{-\rho t}}{n_t x_t(\omega)^{\psi} \int_0^{n_t} x_t(\omega')^{1-\psi} d\omega'} = \mu_{1t} p_t(\omega) \tag{A.1}$$

$$H_{k_t} : \mu_{2t} - (\delta + g_A + g_L) \mu_{1t} = -\dot{\mu}_{1t} \tag{A.2}$$

$$H_{u_t(\omega)} : \mu_{1t} \frac{p_t(\omega)(1-\alpha)(1-\gamma_t(\omega)) x_t(\omega) u_t(\omega)^{\sigma-1}}{\gamma_t(\omega) s_t(\omega)^{\sigma} + (1-\gamma_t(\omega)) u_t(\omega)^{\sigma}} = \mu_{3t} \tag{A.3}$$

$$H_{s_t(\omega)} : \mu_{1t} \frac{p_t(\omega)(1-\alpha)\gamma_t(\omega) x_t(\omega) s_t(\omega)^{\sigma-1}}{\gamma_t(\omega) s_t(\omega)^{\sigma} + (1-\gamma_t(\omega)) u_t(\omega)^{\sigma}} = (\mu_{3t} + \mu_{4t}) \tag{A.4}$$

$$H_{k_t(\omega)} : \mu_{1t} p_t(\omega) \alpha \frac{x_t(\omega)}{k_t(\omega)} = \mu_{2t} \tag{A.5}$$

$$H_{e_t} : \mu_{3t} = \theta \mu_{4t} \tag{A.6}$$

$$H_{u_t(\iota)} : \mu_{1t} \frac{i_t (1-\alpha)(1-\gamma(\iota)) u_t(\iota)^{\sigma-1}}{\gamma(\iota) s_t(\iota)^{\sigma} + (1-\gamma(\iota)) u_t(\iota)^{\sigma}} = \mu_{3t} \tag{A.7}$$

$$H_{s_t(\iota)} : \mu_{1t} \frac{i_t (1-\alpha)\gamma(\iota) s_t(\iota)^{\sigma-1}}{\gamma(\iota) s_t(\iota)^{\sigma} + (1-\gamma(\iota)) u_t(\iota)^{\sigma}} = (\mu_{3t} + \mu_{4t}) \tag{A.8}$$

$$H_{k_t(\iota)} : \mu_{1t} \alpha \frac{i_t}{k_t(\iota)} = \mu_{2t} \tag{A.9}$$

[23] We have added the notation on the left side of the equations to show which consumer first order conditions were used to arrive at each particular equation.

along with the constraints (2), (1), (9), (10), (13), and (12).

It is useful to organize the proof from here into three steps. In the first, we derive the time allocation expressions. In the second, we use these results to find dynamic expressions for capital and capital allocations. In the third we show convergence.

Step 1: We first derive expressions for other labor inputs as functions of $u_t(\omega)$ and technology parameters. Using equation (A.3) for goods ω and ω' and rearranging yields

$$\frac{u_t(\omega')^{1-\sigma}}{u_t(\omega)^{1-\sigma}} = \frac{p_t(\omega')x_t(\omega')}{p_t(\omega)x_t(\omega)} \frac{(1-\gamma_t(\omega'))}{(1-\gamma_t(\omega))} \frac{\gamma_t(\omega)s_t(\omega)^\sigma + (1-\gamma_t(\omega))\, u_t(\omega)^\sigma}{\gamma_t(\omega')s_t(\omega')^\sigma + (1-\gamma_t(\omega'))\, u_t(\omega')^\sigma}. \tag{A.10}$$

Similarly, (A.5) gives

$$\frac{p_t(\omega')\, x_t(\omega')}{p_t(\omega)\, x_t(\omega)} = \frac{k_t(\omega')}{k_t(\omega)}. \tag{A.11}$$

Put (A.11) into (A.10) and simplify to get

$$\frac{u_t(\omega')}{u_t(\omega)} = \frac{k_t(\omega')}{k_t(\omega)} \frac{(1-\gamma_t(\omega'))}{(1-\gamma_t(\omega))} \frac{\gamma_t(\omega)\left(\frac{s_t(\omega)}{u_t(\omega)}\right)^\sigma + (1-\gamma_t(\omega))}{\gamma_t(\omega')\left(\frac{s_t(\omega')}{u_t(\omega')}\right)^\sigma + (1-\gamma_t(\omega'))}. \tag{A.12}$$

Use (A.3), (A.4) and (A.6) to show that

$$\frac{s_t(\omega)}{u_t(\omega)} = \left[\frac{\gamma_t(\omega)}{(1-\gamma_t(\omega))} \frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}}. \tag{A.13}$$

Put this into (A.12) and rearrange to derive

$$\frac{u_t(\omega')}{u_t(\omega)} = \frac{k_t(\omega')}{k_t(\omega)} \left[\frac{1-\gamma_t(\omega')}{1-\gamma_t(\omega)}\right]^{\frac{1}{1-\sigma}} \frac{z_t(\omega)^\sigma}{z_t(\omega')^\sigma} \tag{A.14}$$

where $z_t(\omega)$ is defined in the proposition. Combining (A.13) and (A.14) gives

$$\frac{u_t(\omega')}{s_t(\omega)} = \frac{k_t(\omega')}{k_t(\omega)} \left[\frac{1-\gamma_t(\omega')}{\gamma_t(\omega)} \frac{1+\theta}{\theta}\right]^{\frac{1}{1-\sigma}} \frac{z_t(\omega)^\sigma}{z_t(\omega')^\sigma}.$$

Next (A.3) and (A.7) give

$$\frac{i_t}{p_t(\omega)x_t(\omega)} = \frac{[\gamma(\iota)s_t(\iota)^\sigma + (1-\gamma(\iota))\, u_t(\iota)^\sigma]}{[\gamma_t(\omega)s_t(\omega)^\sigma + (1-\gamma_t(\omega))\, u_t(\omega)^\sigma]} \frac{(1-\gamma_t(\omega))}{(1-\gamma(\iota))} \frac{u_t(\iota)^{1-\sigma}}{u_t(\omega)^{1-\sigma}}$$

while combining (A.5) and (A.9) gives

$$\frac{k_t(\iota)}{k_t(\omega)} = \frac{i_t}{p_t(\omega)\, x_t(\omega)}.$$

Combining the above two equations and simplifying gives

$$\frac{u_t(\iota)}{u_t(\omega)} = \frac{k_t(\iota)}{k_t(\omega)} \left[\frac{1-\gamma(\iota)}{1-\gamma_t(\omega)}\right]^{\frac{1}{1-\sigma}} \frac{z_t(\omega)^\sigma}{z(\iota)^\sigma} \tag{A.15}$$

and similarly

$$\frac{s_t(\iota)}{u_t(\omega)} = \frac{k_t(\iota)}{k_t(\omega)} \left[\frac{\gamma(\iota)}{1-\gamma_t(\omega)} \frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}} \frac{z_t(\omega)^\sigma}{z(\iota)^\sigma}. \tag{A.16}$$

It is necessary to remove the capital expressions from (A.14)-(A.16). Toward this end, note that (A.1) yields

$$\frac{x_t\left(\omega'\right)^{\psi}}{x_t\left(\omega\right)^{\psi}}=\frac{p_t\left(\omega\right)}{p_t\left(\omega'\right)}. \tag{A.17}$$

With (A.11) this is

$$\frac{k_t(\omega')}{k_t(\omega)}=\frac{x_t\left(\omega'\right)^{1-\psi}}{x_t\left(\omega\right)^{1-\psi}}.$$

Using (1) and (A.13)-(A.16) and factoring out $\frac{u_t(\omega')}{u_t(\omega)}$ this can be written as

$$\frac{k_t(\omega')}{k_t(\omega)}=\left[\frac{k_t\left(\omega'\right)}{k_t\left(\omega\right)}\right]^{\alpha(1-\psi)}\left(\frac{u_t\left(\omega'\right)}{u_t\left(\omega\right)}\right)^{(1-\alpha)(1-\psi)}\left[\frac{\gamma_t(\omega')\left[\frac{\gamma_t(\omega')}{(1-\gamma_t(\omega'))}\frac{\theta}{1+\theta}\right]^{\frac{\sigma}{1-\sigma}}+(1-\gamma_t(\omega'))}{\gamma_t(\omega)\left[\frac{\gamma_t(\omega)}{(1-\gamma_t(\omega))}\frac{\theta}{1+\theta}\right]^{\frac{\sigma}{1-\sigma}}+(1-\gamma_t(\omega))}\right]^{\frac{(1-\alpha)(1-\psi)}{\sigma}}.$$

Using (A.12) this simplifies to

$$\frac{k_t(\omega')}{k_t(\omega)}=\left[\frac{z_t(\omega')}{z_t(\omega)}\right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}}. \tag{A.18}$$

Given this expression for capital ratios, equations (A.12) and (A.13) simplify to

$$\frac{u_t\left(\omega'\right)}{u_t\left(\omega\right)}=\left[\frac{z_t(\omega')}{z_t(\omega)}\right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}\left[\frac{1-\gamma_t\left(\omega'\right)}{1-\gamma_t\left(\omega\right)}\right]^{\frac{1}{1-\sigma}} \tag{A.19}$$

and

$$\frac{u_t\left(\omega'\right)}{s_t(\omega)}=\left[\frac{1-\gamma_t\left(\omega'\right)}{\gamma_t\left(\omega\right)}\frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}}\left[\frac{z_t(\omega')}{z_t(\omega)}\right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}. \tag{A.20}$$

We now need to remove the capital expression in (A.15) and (A.16). To do this,, rearrange (A.15) and integrate each side to get

$$\int_0^{n_t}\frac{k_t\left(\omega\right)}{k_t\left(\iota\right)}d\omega=\int_0^{n_t}\frac{u_t(\omega)}{u_t(\iota)}\left[\frac{1-\gamma\left(\iota\right)}{1-\gamma_t\left(\omega\right)}\right]^{\frac{1}{1-\sigma}}\frac{z_t(\omega)^{\sigma}}{z(\iota)^{\sigma}}d\omega$$

or

$$\frac{(1-\nu_t)}{\nu_t}\frac{u_t(\iota)}{(1-\gamma\left(\iota\right))^{\frac{1}{1-\sigma}}}z(\iota)^{\sigma}=\int_0^{n_t}u_t(\omega)\left[\frac{1}{1-\gamma_t\left(\omega\right)}\right]^{\frac{1}{1-\sigma}}z_t(\omega)^{\sigma}d\omega.$$

Substitute in for $u_t(\omega)$ using (A.14) and simplify to get

$$\frac{(1-\nu_t)}{\nu_t}\frac{u_t(\iota)z(\iota)^{\sigma}}{(1-\gamma\left(\iota\right))^{\frac{1}{1-\sigma}}}=\frac{u_t\left(\omega'\right)z_t(\omega')^{\sigma}}{\left[1-\gamma_t\left(\omega'\right)\right]^{\frac{1}{1-\sigma}}}\int_0^{n_t}\frac{k_t\left(\omega\right)}{k_t\left(\omega'\right)}d\omega$$

which can be used to derive

$$u_t(\iota)=\frac{\nu_t}{1-\nu_t}\frac{(1-\gamma\left(\iota\right))^{\frac{1}{1-\sigma}}}{\left[1-\gamma_t\left(\omega\right)\right]^{\frac{1}{1-\sigma}}}\frac{Z_t u_t\left(\omega\right)}{z(\iota)^{\sigma}z_t(\omega)^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}}} \tag{A.21}$$

where Z_t is defined in the proposition. Analogous algebra yields

$$s_t(\iota) = \frac{\nu_t}{1-\nu_t}\left[\frac{\gamma(\iota)}{1-\gamma_t(\omega)}\frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}} \frac{Z_t u_t(\omega)}{z(\iota)^\sigma z_t(\omega)^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}}}. \tag{A.22}$$

We now use the time constraint to solve for $u_t(\omega)$. Since $S_t = \theta e_t$, (13) can be written as

$$1 = \int_0^{n_t} u_t(\omega)\,d\omega + \left(\frac{1+\theta}{\theta}\right)\int_0^{n_t} s_t(\omega)\,d\omega + u_t(\iota) + \left(\frac{1+\theta}{\theta}\right)s_t(\iota).$$

Using (A.19)-(A.22), this constraint gives

$$\begin{aligned}
1 = &\int_0^{n_t}\left[\frac{z_t(\omega)}{z_t(\omega')}\right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}}\left[\frac{1-\gamma_t(\omega)}{1-\gamma_t(\omega')}\right]^{\frac{1}{1-\sigma}} u_t(\omega')\,d\omega\\
&+\left(\frac{1+\theta}{\theta}\right)\int_0^{n_t}\left[\frac{\gamma_t(\omega)}{1-\gamma_t(\omega')}\frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}}\left[\frac{z_t(\omega)}{z_t(\omega')}\right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} u_t(\omega')\,d\omega\\
&+\frac{\nu_t}{1-\nu_t}\frac{(1-\gamma(\iota))^{\frac{1}{1-\sigma}}}{[1-\gamma_t(\omega')]^{\frac{1}{1-\sigma}}}\frac{Z_t}{z_t(\omega')^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} z(\iota)^\sigma} u_t(\omega')\\
&+\left(\frac{1+\theta}{\theta}\right)\frac{\nu_t}{1-\nu_t}\frac{1}{z_t(\omega')^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} z(\iota)^\sigma}\left[\frac{\gamma(\iota)}{1-\gamma_t(\omega')}\frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}} Z_t u_t(\omega')
\end{aligned}$$

or

$$\begin{aligned}
1 = &\int_0^{n_t}\left[\frac{z_t(\omega)}{z_t(\omega')}\right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} u_t(\omega')\\
&\left[\left[\frac{1-\gamma_t(\omega)}{1-\gamma_t(\omega')}\right]^{\frac{1}{1-\sigma}} + \left(\frac{1+\theta}{\theta}\right)\left[\frac{\gamma_t(\omega)}{1-\gamma_t(\omega')}\frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}}\right]d\omega\\
&+\frac{\nu_t}{1-\nu_t}\frac{1}{z_t(\omega')^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}} z(\iota)^\sigma} Z_t u_t(\omega')\frac{1}{[1-\gamma_t(\omega')]^{\frac{1}{1-\sigma}}}\\
&\left[(1-\gamma(\iota))^{\frac{1}{1-\sigma}} + \left(\frac{\theta}{1+\theta}\right)\left[\gamma(\iota)\frac{\theta}{1+\theta}\right]^{\frac{1}{1-\sigma}}\right]
\end{aligned}$$

which can be written as

$$\begin{aligned}
1 = &\frac{u_t(\omega')}{[1-\gamma_t(\omega')]^{\frac{1}{1-\sigma}}}\int_0^{n_t}\left[\frac{z_t(\omega)}{z_t(\omega')}\right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}} z_t(\omega)^\sigma d\omega\\
&+\frac{\nu_t}{1-\nu_t}\frac{1}{z_t(\omega')^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\psi\sigma}{\psi}} z(\iota)^\sigma} Z_t u_t(\omega')\frac{1}{[1-\gamma_t(\omega')]^{\frac{1}{1-\sigma}}} z(\iota)^\sigma
\end{aligned}$$

and finally simplified to

$$1 = u_t(\omega')\frac{Z_t}{z_t(\omega')^{\frac{(1-\alpha)(1-\psi)(1-\sigma)-\sigma\psi}{\psi}}[1-\gamma_t(\omega')]^{\frac{1}{1-\sigma}}}\left[1+\frac{\nu_t}{1-\nu_t}\right].$$

As this relationship holds for any good, we can drop the prime and solve for $u_t(\omega)$ as given in (18). Then substituting this expression into (A.20)-(A.22) gives (19)-(21).

Step 2. We now derive the goods market results. (A.1) will be used to analyze the dynamics. Dynamics are expressed in terms of ν_t, k_t and the underlying parameters. Thus the first task is to solve for $x_t(\omega)$ and $p_t(\omega)$ in terms of these items so that they can be eliminated in (A.1). Using (18) and (19) in (2) and simplifying gives

$$y_t(\omega) = k_t(\omega)^{\alpha} \left(\frac{(1-\nu_t)\, z_t(\omega)^{\frac{(1-\alpha)(1-\psi)(1-\sigma)+(1-\sigma)\psi}{\psi}}}{Z_t} \right)^{1-\alpha}. \tag{A.23}$$

To eliminate $k_t(\omega)$ from this expression, use (A.18) and the definition of ν_t to get

$$(1-\nu_t)\, k_t = \int k_t(\omega)\, d\omega = \int \left[\frac{z_t(\omega)}{z_t(\omega')} \right]^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} k_t(\omega') d\omega$$

which can be solved to get

$$k_t(\omega') = \frac{(1-\nu_t)\, z_t(\omega')^{\frac{(1-\alpha)(1-\psi)(1-\sigma)}{\psi}} k_t}{Z_t}.$$

As this holds for all products, the prime can be dropped. Furthermore, since market clearing requires $y_t(\omega) = x_t(\omega)$, putting the above expression into (A.23) and rearranging gives

$$x_t(\omega) = \frac{z_t(\omega)^{\frac{(1-\alpha)(1-\sigma)}{\psi}} (1-\nu_t)\, k_t^{\alpha}}{Z_t}. \tag{A.24}$$

To solve for prices, note (A.5) and (A.9) give

$$\frac{k_t(\iota)}{k_t(\omega)} = \frac{i_t}{p_t(\omega)\, x_t(\omega)}.$$

Using equations (2) and (1), this is

$$\frac{k_t(\iota)}{k_t(\omega)} = \frac{k_t(\iota)^{\alpha} \left[\gamma(\iota) s_t(\iota)^{\sigma} + (1-\gamma(\iota))\, u_t(\iota)^{\sigma}\right]^{\frac{1-\alpha}{\sigma}}}{p_t(\omega)\, k_t(\omega)^{\alpha} \left[\gamma_t(\omega) s_t(\omega)^{\sigma} + (1-\gamma_t(\omega))\, u_t(\omega)^{\sigma}\right]^{\frac{1-\alpha}{\sigma}}}.$$

Then using equations (18)-(19) in the above expression, simplifying and solving for $p_t(\omega)$ it can be shown that

$$p_t(\omega) = \frac{z(\iota)^{(1-\sigma)(1-\alpha)}}{z_t(\omega)^{(1-\sigma)(1-\alpha)}}. \tag{A.25}$$

We now consider the dynamics. Putting equations (A.24) and (A.25) into (A.1) and simplifying gives

$$\frac{1}{\mu_{1t}} = e^{\rho t} \left[z(\iota)^{(1-\sigma)(1-\alpha)} \right] \left[\frac{(1-\nu_t)\, k_t^{\alpha}}{Z_t} \right] \int_0^{n_t} z(\omega')^{\frac{(1-\alpha)(1-\sigma)(1-\psi)}{\psi}} d\omega'$$

Using the definition of Z_t in the proposition this is

$$\frac{1}{\mu_{1t}} = e^{\rho t} z(\iota)^{(1-\sigma)(1-\alpha)} (1-\nu_t)\, k_t^{\alpha}.$$

Taking the natural log and time differential of each side gives

$$-\frac{\dot{\mu}_{1t}}{\mu_{1t}} = \rho + \frac{\dot{\overline{1-\nu_t}}}{1-\nu_t} + \alpha \frac{\dot{k}_t}{k_t}. \tag{A.26}$$

Using (9) with the goods market clearance conditions, $y_t(\iota) = i_t$, $y_t(\omega) = x_t(\omega)$ for $0 \leq \omega \leq n_t$, as well as the input market prices, (3), (4), (5), gives

$$i_t = \dot{k}_t + (\delta + g_A + g_L)\, k_t. \tag{A.27}$$

Using (A.2), (A.9) and (A.27), (A.26) becomes

$$\alpha \frac{i_t}{k_t(\iota)} - (\delta + g_A + g_L) = \rho + \frac{\dot{\overline{1-\nu_t}}}{1-\nu_t} + \alpha \left[\frac{i_t}{k_t} - (\delta + g_A + g_L) \right]. \tag{A.28}$$

Use (1), (14), (20) and (21)to get

$$i_t = \nu_t k_t^{\alpha} z(\iota)^{(1-\sigma)(1-\alpha)}$$

which implies

$$\frac{i_t}{k_t} = \nu_t k_t^{\alpha-1} z(\iota)^{(1-\sigma)(1-\alpha)} \quad \text{or} \quad \frac{i_t}{k_t(\iota)} = k_t^{\alpha-1} z(\iota)^{(1-\sigma)(1-\alpha)}. \tag{A.29}$$

This can be substituted into (A.28) and rearranged to get (16). To derive (17), use (A.29) to rewrite (A.27).

Step 3. To show global stability, we first show that the steady state is locally a saddle point. We then use a phase diagram to argue that saddle path stability holds more generally. This argument builds on that of Barro and Sala-i-Martin (2004).

Local stability. Rewrite (16) and (17) as

$$\begin{aligned}
\frac{d \ln (1-\nu_t)}{dt} &= \alpha z(\iota)^{(1-\sigma)(1-\alpha)} e^{(\alpha-1)\ln k_t} e^{\ln(1-\nu_t)} - (1-\alpha)(\delta + g_A + g_L) + g_n - \rho \\
\frac{d \ln k_t}{dt} &= z(\iota)^{(1-\sigma)(1-\alpha)} e^{(\alpha-1)\ln k_t} - z(\iota)^{(1-\sigma)(1-\alpha)} e^{(\alpha-1)\ln k_t} e^{\ln(1-\nu_t)} - (\delta + g_A + g_L).
\end{aligned}$$

Let k and ν (absent time subscripts) be steady state values; i.e. values such that $\dot{\overline{(1-\nu_t)}} = \dot{k}_t = 0$. Straightforward calculation show that

$$\begin{aligned}
(1-\nu) &= \frac{(1-\alpha)(\delta + g_A + g_L) - g_n + \rho}{\delta + g_A + g_L - g_n + \rho} \\
k &= \left(\frac{\delta + g_A + g_L - g_n + \rho}{\alpha z(\iota)^{(1-\sigma)(1-\alpha)}} \right)^{\frac{1}{\alpha-1}}.
\end{aligned} \tag{A.30}$$

Taking a first-order Taylor series expansion of (A.30) around the steady state values gives

$$\begin{bmatrix} \frac{d \ln(1-\nu_t)}{dt} \\ \frac{d \ln k_t}{dt} \end{bmatrix} = \begin{bmatrix} Q & -(1-\alpha) Q \\ -(1-\alpha)(\delta + g_A + g_L) & -\frac{1}{\alpha} Q \end{bmatrix} \begin{bmatrix} \ln \frac{1-\nu_t}{1-\nu} \\ \ln \frac{k_t}{k} \end{bmatrix}$$

where $Q \equiv (1-\alpha)(\delta + g_A + g_L) - g_n + \rho > 0$. It is clear from this expression that the determinant of the characteristic matrix is negative. This implies that the eigenvalues of the system are of opposite signs which implies local saddle-path stability.

The phase diagram. Figure 4 presents the diagram. The upward sloping curve corresponds to the $\dot{\overline{(1-\nu_t)}} = 0$ condition. Equation (16) shows that for values of k_t to the left of this curve, $1-\nu_t$ is rising and that for values to the right, $1-\nu_t$ is falling. The downward sloping curve corresponds to the $\frac{\dot{k}_t}{k_t} = 0$ condition. From equation (17), k_t is decreasing for values of $1-\nu_t$ to the right of this curve and rising for values to the left. At the point of intersection of these curves, $\nu_t = \nu$ and $k_t = k$ and the economy is in balanced growth.

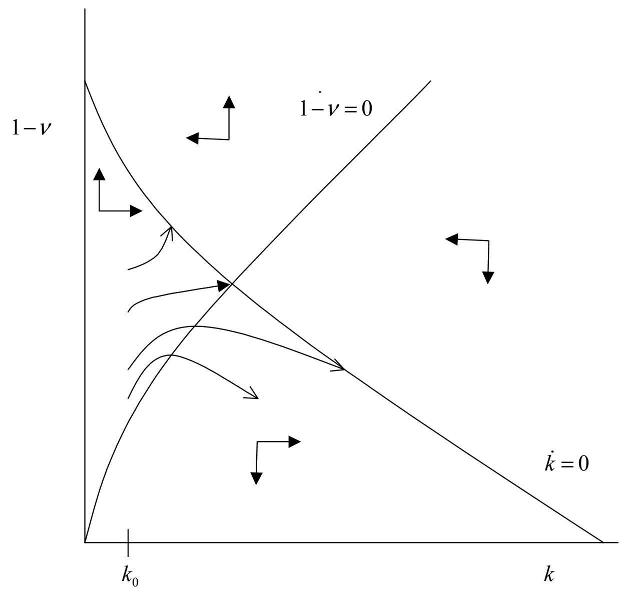

Figure 4: **The phase diagram.** The arrows show the direction of motion for $1-\nu$ and k in each of the four regions defined by the $\dot{1-\nu}=0$ and $\dot{k}=0$ loci. The path with a solid arrow head is the saddle-point stable path.

These curves divide the positive k, $1-\nu$ space into 4 regions. In the region to the left of $\dot{(1-\nu_t)}=0$ and to the right of $\dot{k}_t=0$, the savings rate is too low and $1-\nu_t$ rises and k_t falls as indicated by the upward and leftward arrows. In this region movement is away from $\nu_t=\nu$ and $k_t=k$. Assuming investment is reversible, in finite time the vertical axis is hit and the capital stock goes to 0.[24] With a zero capital stock (2) implies $x_t(\omega)=0$ in finite time. Next note that substituting (A.17) into (A.1) implies that $e^{-\rho t}=\mu_{1t}[p_t(\omega)]^{\frac{1}{\psi}}n_t x_t(\omega)\int_0^{n_t}p_t(\omega')^{\frac{\psi-1}{\psi}}d\omega'$. As is shown in (A.35) below, prices are bounded. Then so long as μ_{1t} is bounded, putting $x_t(\omega)=0$ into this equation implies $e^{-\rho t}=0$ which cannot hold in finite time. To see that μ_{1t} is bounded, note that equations (A.2), (A.9), (14), $k_t(\iota)=\nu_t k_t$ and $i_t=z(\iota)^{(1-\sigma)(1-\alpha)}\nu_t k_t^{\alpha}$ give

$$\frac{\dot{\mu}_{1t}}{\mu_{1t}}=(\delta+g_A+g_L)-\alpha z(\iota)^{(1-\sigma)(1-\alpha)}k_t^{\alpha-1}$$

so that

$$\mu_{1t}=\mu_{10}\exp\left[\int_0^t\left[(\delta+g_A+g_L)-\alpha z(\iota)^{(1-\sigma)(1-\alpha)}k_t^{\alpha-1}\right]d\mu_{1t}\right]. \tag{A.31}$$

where μ_{10} is finite and positive. As k_t approaches 0, the integral goes to negative infinity so μ_{1t} goes to 0.

In the region to the right of $\dot{(1-\nu_t)}=0$ and to the left of $\dot{k}_t=0$, savings is too high and $1-\nu_t$ falls and k_t increases as indicated by the downward and rightward arrows in this region. In this region movement is again away from $\nu_t=\nu$ and $k_t=k$. The value of $1-\nu_t$ is bounded by 0 but if this is reached, the rightward movement continues. Eventually the economy goes to the point where the $\dot{k}_t=0$ intersects with the horizontal axis. Here, the capital stock is constant and ν_t is

[24] The case of irreversible investment also holds and the argument is analogous to that in Barro and Sala-i-Martin (2004) pages 134-135.

constant at 1. However, this equilibrium violates the transversality condition. To see this, putting equation (A.31) into the transversality condition gives

$$\lim_{t\to\infty}\left\{k_t\mu_{10}\exp\left[-\int_0^t\left[\alpha z(\iota)^{(1-\sigma)(1-\alpha)}k_t^{\alpha-1}-(\delta+g_A+g_L)\right]d\mu_{1t}\right]\right\}=0$$

Since k_t is positive and finite in this equilibrium, this requires that $\alpha z(\iota)^{(1-\sigma)(1-\alpha)}k_t^{\alpha-1} > \delta+g_A+g_L$. With $\dot{k}_t = 0$ and $\nu_t = 1$ equation (17) can be used to show that $z(\iota)^{(1-\sigma)(1-\alpha)}k_t^{\alpha-1} = \delta+g_A+g_L$. Thus after some simplification, the requirement becomes $\alpha > 1$ which cannot hold by assumption. Thus a stable arm is possible only the regions to the left of each curve or to the right of each curve. The arrows demonstrate that in these regions, movement is in the direction of $\nu_t = \nu$ and $k_t = k$.

We now demonstrate that the initial choice of $1-\nu_0$ puts the economy on the stable arm. Consider the case where the initial capital stock, k_0, is less than k. The stable arm is the one leading to the the point where $(\dot{1-\nu_t}) = \dot{k}_t = 0$. Suppose $1-\nu_0$ is chosen to be above the level associated with this stable arm. Then eventually, the path crosses $\dot{k}_t = 0$ curve and enters the region to the left of $(\dot{1-\nu_t}) = 0$ and to the right of $\dot{k}_t = 0$ which was shown above to be suboptimal. Next, suppose $1-\nu_0$ is chosen to be below the level associated with this stable arm. Then eventually, the path crosses the $(\dot{1-\nu_t})$ curve and enters the region to the right of $(\dot{1-\nu_t}) = 0$ and to the left of $\dot{k}_t = 0$, which again is suboptimal. Thus the optimal choice of $1-\nu_0$ puts the economy on the stable arm. A symmetric argument holds if initially $k_0 > k$.

A.2 Proof of Corollary 1

With $\psi = 1,\ Z_t \equiv n$. Let ν be the steady state value of ν_t. Given this and $\sigma = 0$, (18)-(21) are

$$\begin{aligned}
u_t(\omega) &= (1-\nu)\frac{1-\gamma_t(\omega)}{n_t}\\
s_t(\omega) &= \frac{\theta}{1+\theta}\frac{\gamma_t(\omega)(1-\nu)}{n_t}\\
u_t(\iota) &= \nu(1-\gamma(\iota))\\
s_t(\iota) &= \nu\frac{\gamma(\iota)\theta}{1+\theta}
\end{aligned}$$

Given this, integration over the product space gives

$$U_t = u_t(\iota) + \int_0^{n_t} u_t(\omega)d\omega = (1-\nu)(1-\widetilde{\gamma}_t) + \nu(1-\gamma(\iota)), \tag{A.32}$$

$$S_t = s_t(\iota) + \int_0^{n_t} s_t(\omega)d\omega = \frac{\theta}{1+\theta}((1-\nu)\widetilde{\gamma}_t + \nu\gamma(\iota)), \tag{A.33}$$

and

$$e_t = \frac{S_t}{\theta} = \frac{1}{1+\theta}((1-\nu)\widetilde{\gamma}_t + \nu\gamma(\iota)). \tag{A.34}$$

The corollary follows directly from these three expressions.

A.3 Proof of Corollary 2a

Item (a) is immediate from (A.32)-(A.34). Next, using L'Hopital's rule to define $z(\iota)$ and $z_t(\omega)$ at $\sigma = 0$ and putting these expressions into (A.25) gives

$$p_t(\omega) = \frac{\left(\left(\gamma(\iota)\frac{\theta}{1+\theta}\right)^{\gamma(\iota)}(1-\gamma(\iota))^{(1-\gamma(\iota))}\right)^{1-\alpha}}{\left(\left(\gamma_t(\omega)\frac{\theta}{1+\theta}\right)^{\gamma_t(\omega)}(1-\gamma_t(\omega))^{(1-\gamma_t(\omega))}\right)^{1-\alpha}}. \tag{A.35}$$

Notice that the price of the consumption good is increasing only if $\left(\gamma_t(\omega)\frac{\theta}{1+\theta}\right)^{\gamma(\iota)}(1-\gamma_t(\omega))^{(1-\gamma_t(\omega))}$ is decreasing, i.e. if $d\frac{\left(\tilde{\gamma}_t\frac{\theta}{1+\theta}\right)^{\tilde{\gamma}_t}((1-\tilde{\gamma}_t))^{1-\tilde{\gamma}_t}}{dt} < 0$. This has the same sign as

$$d\frac{\tilde{\gamma}_t\ln\tilde{\gamma}_t + \tilde{\gamma}_t\ln\frac{\theta}{1+\theta} + (1-\tilde{\gamma}_t)\ln(1-\tilde{\gamma}_t)}{dt} = \left(\ln\tilde{\gamma}_t + \ln\frac{\theta}{1+\theta} - \ln(1-\tilde{\gamma}_t)\right)\dot{\tilde{\gamma}}_t$$

and is negative if $\ln\tilde{\gamma}_t + \ln\frac{\theta}{1+\theta} - \ln(1-\tilde{\gamma}_t) < 0$ or $\tilde{\gamma}_t\frac{\theta}{1+\theta} < 1-\tilde{\gamma}_t$. That is, if $\tilde{\gamma}_t < \frac{\theta+1}{2\theta+1}$.

A.4 Proof of Corollary 2b

To see (a) note $\int\ln(\omega+1)\,d\omega = (\omega+1)\ln(\omega+1) - (\omega+1)$. Thus

$$\int_0^{n_t}\ln(\omega+1)\,d\omega = (n_t+1)\ln(n_t+1) - n_t.$$

Putting this into equation 29 gives

$$\widetilde{\gamma}_t = \frac{1}{n_t}\int_0^{n_t}\gamma_t(\omega)d\omega = \underline{\gamma} + (\overline{\gamma}-\underline{\gamma})\left(\frac{n_t+1}{n_t} - \frac{1}{\ln(n_t+1)}\right).$$

From this it is clear that $\lim_{t\longrightarrow\infty}\widetilde{\gamma}_t = \lim_{n_t\longrightarrow\infty}\widetilde{\gamma}_t = \overline{\gamma}$. It remains to show that

$$\dot{\widetilde{\gamma}}_t = (\overline{\gamma}-\underline{\gamma})\left(-\frac{1}{n_t^2} + \frac{1}{(n_t+1)(\ln(n_t+1))^2}\right)\dot{n}_t > 0$$

This requires $n_t^2 > (n_t+1)(\ln(n_t+1))^2$ or $n_t(n_t+1)^{-\frac{1}{2}} > \ln(n_t+1)$. With $n_t = 0$, $\frac{n_t}{(n_t+1)^{\frac{1}{2}}} = \ln(n_t+1)$. Thus if $\frac{\partial n_t(n_t+1)^{-\frac{1}{2}}}{\partial n_t} > \frac{\partial\ln(n_t+1)}{\partial n_t}$ the condition is met. Upon differentiating and simplifying this requires $(n_t+1)^{\frac{1}{2}} - \frac{n_t}{2}(n_t+1)^{-\frac{1}{2}} > 1$ or $n_t+2 > 2(n_t+1)^{\frac{1}{2}}$. With $n_t = 0$, $n_t+2 = 2(n_t+1)^{\frac{1}{2}}$. Thus if $\frac{\partial(n_t+2)}{\partial n_t} > \frac{\partial 2(n_t+1)^{\frac{1}{2}}}{\partial n_t}$ the condition is met. This requires $1 > \frac{1}{(n_t+1)^{\frac{1}{2}}}$ which holds for positive n_t. Item (b) is immediate from the total labor relationships in part (b) of the proposition.

A.5 Proof of Corollary 2c

To see (a), note

$$\int_0^{n_t}\gamma_t(\omega)d\omega = \left(\underline{\gamma}\omega_t + (\overline{\gamma}-\underline{\gamma})\frac{\omega_t^{\rho+1}}{1+\omega_t^{\rho}}\right)\Bigg|_0^{n_t}.$$

Differentiate to verify. This gives

$$\widetilde{\gamma}_t = \frac{1}{n_t}\int_0^{n_t} \gamma_t(\omega)d\omega = \underline{\gamma} + \left(\overline{\gamma} - \underline{\gamma}\right)\frac{n_t^{\rho}}{1+n_t^{\rho}}.$$

Both $\lim_{t\longrightarrow\infty} \widetilde{\gamma}_t = \lim_{n_t\longrightarrow\infty} \widetilde{\gamma}_t = \overline{\gamma}$ and $\dot{\widetilde{\gamma}}_t > 0$ are immediate from this expression. Item (b) is immediate from the total labor relationships in part (b).

B Appendix B: Data

The data plotted in Figures 1a, 1b and 2 came from the following sources:

The real GDP and real investment series, published by the U.S. Department of Commerce, are measured in billions of 2000 chained dollars and can be found online at http://research.stlouisfed.org/fred2/series/GDPCA/106 and http://research.stlouisfed.org/fred2/series/GPDICA/112 respectively. The real net physical capital stock series, published by the U.S. Department of Commerce, is measured in billions of 1992 chained dollars and can be found online at, http://www.bea.doc.gov/bea/an/0597niw/table15.htm. The real interest rate was computed from the 10 year constant maturity nominal interest rate, published by the Board of Governors of the Federal Reserve System and available online at http://research.stlouisfed.org/fred2/series/GS10/22, and the consumer price index for all urban consumers and including all items, published by the U.S. Department of Labor and available online at http://research.stlouisfed.org/fred2/series/CPIAUCSL/9. The labor share calculation used the compensation of employees and proprietor's income with inventory valuation and capital consumption adjustment series published by U.S. Department of Commerce and available at http://www.bea.doc.gov/bea/dn/nipaweb/TableView.asp#Mid along with the nominal GDP series published by the U.S. Department of Commerce available at http://research.stlouisfed.org/fred2/series/GDP/106. The series for the percent of the population with high school and college degrees, published by the U.S. Census Bureau can be found online at. http://www.census.gov/population/socdemo/education/tabA-2.xls and the series on the percent of 20-21 year olds enrolled in school, published by the U.S. Census Bureau can be found online at http://www.census.gov/population/socdemo/school/tabA-2.xls.

Years with missing data were linearly interpolated.

C Technical appendix (Not intended for publication)

This appendix provides several small demonstrations noted in the paper. Because they are not essential to the primary objective of the paper, they are included in this separate appendix not intended for publication.

C.1 Converting from aggregate form into intensive form

We begin by specifying the model in aggregate form and converting it into the intensive form. The demonstration is carried out for the social planning version of the model. The steps are analogous for the competitive version used in the paper. As in the paper we use a parenthetic ω to index the consumption good space where $\omega \in [0, n_t]$ and n_t is the number of consumer goods available at time t and we use the subscript t to indicate the date.

Suppose that the aggregate production functions for the capital good and consumer good ω at time t are given by

$$Y_t(\iota) = K_t(\iota)^{\alpha} \left[A_t \left[\gamma(\iota) S_t(\iota)^{\sigma} + (1 - \gamma(\iota)) U_t(\iota)^{\sigma} \right]^{\frac{1}{\sigma}} \right]^{1-\alpha},$$

$$Y_t(\omega) = K_t(\omega)^{\alpha} \left[A_t \left[\gamma_t(\omega) S_t(\omega)^{\sigma} + (1 - \gamma_t(\omega)) U_t(\omega)^{\sigma} \right]^{\frac{1}{\sigma}} \right]^{1-\alpha},$$

where the model parameters are such that $0 < \alpha < 1$, $\sigma \leq 1$, $0 \leq \gamma(\iota) \leq 1$ and $0 \leq \gamma_t(\omega) \leq 1$. We let $Y_t(\iota)$ and $Y_t(\omega)$ denote capital good output and consumption good ω output at time t. Similarly, $K_t(\iota)$ and $K_t(\omega)$ denote the capital input used in the production of the various products at time t, A_t denotes an index for the level of technology at time t, $S_t(\iota)$ and $S_t(\omega)$ denote the number of units of skilled labor used in the production of the various products at time t and $U_t(\iota)$ and $U_t(\omega)$ denote the number of units of unskilled labor used in the production the various products ω at time t. Although it is possible to specify A_t as growing due to an endogenous process, for this paper we simply specify that it grows at an exogenous rate of g_A. Adding an endogenous structure for A_t will not change our qualitative point that balanced growth in the goods sector is consistent with labor trends.

The aggregate capital stock is allocated between the various product production process such that

$$K_t = K_t(\iota) + \int_0^{n_t} K_t(\omega)\, d\omega.$$

In addition, we assume that the aggregate capital stock evolves according to

$$\dot{K}_t = Y_t(\iota) - \delta K_t \tag{B.1}$$

where $0 < \delta < 1$ is the depreciation rate for capital. Equation (B.1) shows that capital changes are due to, 1) production of the investment good and 2) the portion of the capital stock that depreciates each period given by δK_t.

We assume that the population grows are rate g_L. At time t there are L_t agents each with one unit of time which can be allocated to unskilled work, skilled work or education. Using notation described above with the addition of E_t for the amount of time spent acquiring skills, the time allocation constraint is given by

$$U_t(\iota) + \int_0^{n_t} U_t(\omega) d\omega + S_t(\iota) + \int_0^{n_t} S_t(\omega) d\omega + E_t = L_t.$$

To convert to intensive form we divide through the goods conditions by $A_t L_t$ to get

$$\frac{Y_t(\iota)}{A_t L_t} = \left(\frac{K_t(\iota)}{A_t L_t} \right)^{\alpha} \left[\left[\gamma(\iota) \left(\frac{S_t(\iota)}{L_t} \right)^{\sigma} + (1 - \gamma(\iota)) \left(\frac{U_t(\iota)}{L_t} \right)^{\sigma} \right]^{\frac{1}{\sigma}} \right]^{1-\alpha},$$

$$\frac{Y_t(\omega)}{A_t L_t} = \left(\frac{K_t(\omega)}{A_t L_t} \right)^{\alpha} \left[\left[\gamma_t(\omega) \left(\frac{S_t(\omega)}{L_t} \right)^{\sigma} + (1 - \gamma_t(\omega)) \left(\frac{U_t(\omega)}{L_t} \right)^{\sigma} \right]^{\frac{1}{\sigma}} \right]^{1-\alpha},$$

and

$$\frac{K_t}{A_t L_t} = \frac{K_t(\iota)}{A_t L_t} + \int_0^{n_t} \frac{K_t(\omega)}{A_t L_t} d\omega.$$

Next we divide the time constraint by L_t

$$\frac{U_t(\iota)}{L_t} + \int_0^{n_t} \frac{U_t(\omega)}{L_t} d\omega + \frac{S_t(\iota)}{L_t} \int_0^{n_t} \frac{S_t(\omega)}{L_t} d\omega + \frac{E_t}{L_t} = 1.$$

Also noting that

$$\begin{aligned}
\left(\frac{\dot{K_t}}{A_t L_t}\right) &= \frac{A_t L_t \dot{K_t} - K_t \left(A_t \dot{L_t} + L_t \dot{A_t}\right)}{(A_t L_t)^2} \\
&= \frac{\dot{K_t}}{A_t L_t} - \frac{K_t}{A_t L_t} \left(\frac{\dot{L_t}}{L_t} + \frac{\dot{A_t}}{A_t}\right) \\
&= \frac{Y_t(\iota) - \delta K_t}{A_t L_t} - \frac{K_t}{A_t L_t} \left(\frac{\dot{L_t}}{L_t} + \frac{\dot{A_t}}{A_t}\right).
\end{aligned}$$

Now defining the intensive form variables $y_t(\iota) \equiv \frac{Y_t(\iota)}{A_t L_t}$, $y_t(\omega) \equiv \frac{Y_t(\omega)}{A_t L_t}$, $k_t \equiv \frac{K_t}{A_t L_t}$, $k_t(\iota) \equiv \frac{K_t(\iota)}{A_t L_t}$, $k_t(\omega) \equiv \frac{K_t(\omega)}{A_t L_t}$, $u_t(\iota) \equiv \frac{U_t(\iota)}{L_t}$, $u_t(\omega) \equiv \frac{U_t(\omega)}{L_t}$, $s_t(\iota) \equiv \frac{S_t(\iota)}{L_t}$, $s_t(\omega) \equiv \frac{S_t(\omega)}{L_t}$ and $e_t \equiv \frac{E_t}{L_t}$, the intensive forms can be written as

$$y_t(\iota) = k_t(\iota)^\alpha \left[\gamma(\iota) s_t(\iota)^\sigma + (1 - \gamma(\iota)) u_t(\iota)^\sigma\right]^{\frac{1-\alpha}{\sigma}}$$

$$y_t(\omega) = k_t(\omega)^\alpha \left[\gamma_t(\omega) s_t(\omega)^\sigma + (1 - \gamma_t(\omega)) u_t(\omega)^\sigma\right]^{\frac{1-\alpha}{\sigma}}$$

$$k_t = k_t(\iota) + \int_0^{n_t} k_t(\omega) d\omega,$$

$$\dot{k_t} = y_t(\iota) - (\delta + g_A + g_L) k_t,$$

and

$$u_t(\iota) + \int_0^{n_t} u_t(\omega) d\omega + s_t(\iota) + \int_0^{n_t} s_t(\omega) d\omega + e_t = 1.$$

Next we focus on the objective function. The aggregate objective is to maximize the discounted value of per capita consumption

$$\int_0^\infty e^{-\rho t} \left[\int_0^{n_t} \ln\left(\frac{C_t(\omega)}{L_t}\right) d\omega\right] dt.$$

where $0 < \rho$ and C_t is an index of current consumption. This index is a CES combination of all goods consumed in period t, i.e.,

$$\frac{C_t}{L_t} \equiv \left[\frac{1}{n_t} \int_0^{n_t} \left(\frac{X_t(\omega)}{L_t}\right)^{1-\psi} d\omega\right]^{\frac{1}{1-\psi}}.$$

where $\psi > 0$. It can be shown that as ψ approaches 1, this converges to

$$\frac{C_t}{L_t} \equiv \exp\left[\frac{1}{n_t} \int_0^{n_t} \ln\left(\frac{X_t(\omega)}{L_t}\right) d\omega\right].$$

Defining the intensive form consumption level for good ω as $x_t(\omega) \equiv \frac{X_t(\omega)}{A_t L_t}$ and defining $c_t \equiv \frac{C_t}{A_t L_t}$, the utility function can be written as

$$\int_0^\infty e^{-\rho t} \ln(A_t c_t(\omega)) dt$$

or setting without loss of generality gives

$$\int_0^\infty e^{-\rho t} \ln(A_0 e^{t g_A} c_t(\omega) dt$$

or

$$\int_0^\infty e^{-\rho t} \ln(e^{t g_A}) dt + \int_0^\infty e^{-\rho t} \ln(c_t(\omega)) dt$$

which is just a monotonic transformation of

$$\int_0^\infty e^{-\rho t} \ln(c_t(\omega)) dt.$$

Furthermore

$$A_t c_t \equiv \left[\frac{1}{n_t} \int_0^{n_t} (A_t x_t(\omega))^{1-\psi} \, d\omega \right]^{\frac{1}{1-\psi}}$$

or

$$c_t \equiv \left[\frac{1}{n_t} \int_0^{n_t} (x_t(\omega))^{1-\psi} \, d\omega \right]^{\frac{1}{1-\psi}}$$

It remains to show that as ψ approaches 1 this converges to

$$c_t \equiv \exp\left[\frac{1}{n_t} \int_0^{n_t} \ln x_t(\omega) d\omega \right].$$

To show this write

$$\ln c_t \equiv \frac{\left[\ln \left[\int_0^{n_t} x_t(\omega)^{1-\psi} d\omega\right] - \ln n\right]}{1-\psi} = \frac{m(\psi)}{n(\psi)}$$

Note that as ψ approaches 1 $m(\psi)$ approaches 0 and $n(\psi)$ approaches 0. Thus L'Hopital's rule can be used to find the limit of $\ln c_t$. Since $c_t = \exp(\ln c_t)$, $\lim c_t \equiv \exp[\lim(\ln c_t)]$. First find $m'(\psi)$

$$m'(\psi) = \frac{\int_0^{n_t} \frac{\partial x_t(\omega)^{1-\psi} d\omega}{\partial \psi} d\omega}{\int_0^{n_t} x_t(\omega)^{1-\psi} d\omega}$$

Recall $\frac{d}{dt} b^t = b^t \ln b$ so

$$m'(\psi) = -\frac{\int_0^{n_t} x_t(\omega)^{1-\psi} \ln[x_t(\omega)] \, d\omega}{\int_0^{n_t} x_t(\omega)^{1-\psi} d\omega}$$

at $(1 = \psi)$ this is

$$m'(\psi) = -\frac{1}{n} \int_0^{n_t} \ln x_t(\omega) d\omega$$

Clearly $n'(\psi) = -1$. Thus

$$\lim c_t \equiv \exp\left[\frac{1}{n} \int_0^{n_t} \ln x_t(\omega) d\omega \right].$$

C.2 Generalized intertemporal elasticity of substitution.

Now let our utility function be

$$U=\int_0^\infty e^{-\rho t}\frac{c_t^{1-\lambda}-1}{1-\lambda}dt$$

where $\lambda>0$ and where (again)

$$c_t\equiv\left[\frac{1}{n_t}\int_0^{n_t}x_t(\omega)^{1-\psi}d\omega\right]^{\frac{1}{1-\psi}}$$

The Hamiltonian is identical except for the first line which becomes

$$H=e^{-\rho t}\frac{\left[\frac{1}{n_t}\int_0^{n_t}x_t(\omega)^{1-\psi}d\omega\right]^{\frac{1-\lambda}{1-\psi}}-1}{1-\lambda}$$

and the first order conditions are identical except for the first which becomes

$$H_{x_t(\omega)}:e^{-\rho t}\frac{x_t\left(\omega\right)^{-\psi}}{n_t}\left[\frac{1}{n_t}\int_0^{n_t}x_t(\omega')^{1-\psi}d\omega'\right]^{\frac{\psi-\lambda}{1-\psi}}=\mu_{1t}p_t\left(\omega\right). \tag{B.2}$$

The proof is identical though equation (A.29). We now derive the goods market results. Equation (B.2) gives

$$\frac{1}{\mu_{1t}}=e^{\rho t}p_t\left(\omega\right)x_t^{\psi}\left(\omega\right)\left[\frac{1}{n_t}\int_0^{n_t}x_t(\omega')^{1-\psi}d\omega'\right]^{\frac{\lambda-\psi}{1-\psi}}n_t$$

Using 27 and 26, this is

$$\frac{1}{\mu_{1t}}=e^{\rho t}\left[\frac{z(\iota)^{(1-\sigma)(1-\alpha)}}{z_t(\omega)^{(1-\sigma)(1-\alpha)}}\right]\left[\frac{z_t(\omega)^{\frac{(1-\alpha)(1-\sigma)}{\psi}}\left(1-\nu_t\right)k_t^{\alpha}}{Z_t}\right]^{\psi}$$
$$\left[\frac{1}{n_t}\int_0^{n_t}\left[\frac{z_t(\omega)^{\frac{(1-\alpha)(1-\sigma)}{\psi}}\left(1-\nu_t\right)k_t^{\alpha}}{Z_t}\right]^{1-\psi}d\omega\right]^{\frac{\lambda-\psi}{1-\psi}}n_t$$

which simplifies to

$$\frac{1}{\mu_{1t}}=e^{\rho t}\left[z(\iota)^{(1-\sigma)(1-\alpha)}\right]\left[\frac{\left(1-\nu_t\right)k_t^{\alpha}}{Z_t}\right]^{\lambda}\left[\int_0^{n_t}z_t(\omega)^{\frac{(1-\alpha)(1-\sigma)(1-\psi)}{\psi}}d\omega\right]^{\frac{\lambda-\psi}{1-\psi}}n_t^{\frac{1-\lambda}{1-\psi}}$$

or

$$\frac{1}{\mu_{1t}}=e^{\rho t}\left[z(\iota)^{(1-\sigma)(1-\alpha)}\right]\left[\left(1-\nu_t\right)k_t^{\alpha}\right]^{\lambda}Z_t^{\frac{-\psi(1-\lambda)}{1-\psi}}n_t^{\frac{1-\lambda}{1-\psi}}.$$

Log and differentiate to get

$$-\frac{\dot{\mu}_{1t}}{\mu_{1t}}=\rho+\lambda\frac{\dot{\overline{1-\nu_t}}}{1-\nu_t}+\lambda\alpha\frac{\dot{k}_t}{k_t}+\frac{1-\lambda}{1-\psi}g_n-\frac{\psi\left(1-\lambda\right)}{1-\psi}\frac{\dot{Z}_t}{Z_t}$$

Equations (A2) and (A9) give

$$\alpha\frac{i_t}{k_t\left(\iota\right)}-\left(\delta+g_A+g_L\right)=\frac{-\dot{\mu}_{1t}}{\mu_{1t}}$$

so

$$\alpha\frac{i_t}{k_t(\iota)} - (\delta + g_A + g_L) = \rho + \lambda\frac{\dot{\overline{1-\nu_t}}}{1-\nu_t} + \lambda\alpha\frac{\dot{k}_t}{k_t} + \frac{1-\lambda}{1-\psi}g_n - \frac{\psi(1-\lambda)}{1-\psi}\frac{\dot{Z}_t}{Z_t}.$$

Our capital accumulation function is $\frac{i_t}{k_t} = \frac{\dot{k}_t}{k_t} + (\delta + g_A + g_L)$ giving

$$\alpha\frac{i_t}{k_t(\iota)} - (\delta + g_A + g_L) = \rho + \lambda\frac{\dot{\overline{1-\nu_t}}}{1-\nu_t} + \lambda\alpha\left[\frac{i_t}{k_t} - (\delta + g_A + g_L)\right] + \frac{1-\lambda}{1-\psi}g_n - \frac{\psi(1-\lambda)}{1-\psi}\frac{\dot{Z}_t}{Z_t}$$

Since $i_t = \nu_t k_t^{\alpha} z(\iota)^{(1-\sigma)(1-\alpha)}$ we know

$$\frac{i_t}{k_t} = \nu_t k_t^{\alpha-1} z(\iota)^{(1-\sigma)(1-\alpha)} \quad \text{or} \quad \frac{i_t}{k_t(\iota)} = k_t^{\alpha-1} z(\iota)^{(1-\sigma)(1-\alpha)}$$

which can be substituted into above. Then upon simplifying and rearranging

$$\frac{\dot{\overline{1-\nu_t}}}{1-\nu_t} = \frac{1}{\lambda}\left[\alpha k_t^{\alpha-1} z(\iota)^{(1-\sigma)(1-\alpha)}(1-\lambda\upsilon_t) - (1-\lambda\alpha)(\delta + g_A + g_L) - \rho - \frac{1-\lambda}{1-\psi}g_n + \frac{\psi(1-\lambda)}{1-\psi}\frac{\dot{Z}_t}{Z_t}\right].$$

This replaces equation (16) while the other relevant differential equation is still give by (17). It is the $\frac{\psi(1-\lambda)}{1-\psi}\frac{\dot{Z}_t}{Z_t}$ term that causes problems. Balanced growth requires that this term be constant. Thus setting $\lambda = 1$ works. This is the logarithmic case. Setting ψ also works but it is easy to show that in this case (perfect substitution) only one good is produced in equilibrium. Finally it is possible that $\frac{\dot{Z}_t}{Z_t} = 0$. However with the underlying z terms changing, this would be a special case. It can be shown that this holds even when $\sigma = 0$.

C.3 Corollary 1 with a non-linear production function for skill

We verify that Corollary 1 continues to hold with this generalization. With this generalization, equation (A.31) becomes

$$S_t = \frac{\eta\theta e_t^{\eta-1}}{1+\theta e_t^{\eta-1}}\left((1-\nu)\widetilde{\gamma}_t + \nu\gamma(\iota)\right)$$

or

$$\theta e_t^{\eta} - \frac{\eta\theta e_t^{\eta-1}}{1+\theta e_t^{\eta-1}}\left((1-\nu)\widetilde{\gamma}_t + \nu\gamma(\iota)\right) = 0$$

Multiplying each side by $\left(1+\eta\theta e_t^{\eta-1}\right)e_t^{1-\eta}\theta^{-1}$ and simplifying gives

$$\eta\left((1-\nu)\widetilde{\gamma}_t + \nu\gamma(\iota)\right) - e_t - \eta\theta e_t^{\eta} = 0.$$

Using the implicit function theorem $\frac{\partial e_t}{\partial\widetilde{\gamma}_t} = \frac{\eta(1-\nu)}{1+\eta^2\theta e_t^{\eta-1}} > 0$. Since the supply of skill increases with education, it also increases with $\widetilde{\gamma}_t$. Convergence to the bounds is assured by the fact that the supply of both skilled and unskilled labor is bounded by 0 and 1 by definition.