Appendix 3: Calibration and Steady State Values (Not intended for publication)

In this appendix, we show how to calibrate the model and find steady state values of all endogenous items. The structure follows the Matlab program 'calibrate_main' available from the authors with equations in the program appearing in the same order as in this appendix. Additionally, this appendix shows the derivation of each equation used in the program and the method of calibrating items where needed. Some of this is repeated from the core paper to make this appendix easier to navigate.

Our strategy for calibrating and solving for steady state values requires iterating on a guess of $\frac{I}{Y}$ until the capital market clears as required by Equation (14). This is discussed in the final subsection of this appendix. For now, we assume $\frac{I}{Y}$ is known and set it equal to .23, which we later find to clear the capital market. The lack of time subscripts throughout indicates steady state values. We set $r = .038$, which is the average value from 1961 to 2018 reported by the World Bank and α=1/3, as is common in the literature. In a steady state $\phi_y = 1$ by definition. For our baseline model we consider log preferences ($\theta = 1$) as is common and set $\tau_\ell = \tau_s = \tau$. The program is written to allow these tax rates to differ but the effects of changing this are modest. Other calibrated items are discussed as they arise in the calculations below.

A3.1 Depreciation

We use these calibrated values to calculate δ_k. From the production function, $\frac{Y}{L} = \left(\frac{K}{L}\right)^\alpha$ and from the law of motion for capital, Equation (9), $\frac{I}{L} = \delta_k \frac{K}{L}$. Together these give

$$\frac{\frac{I}{L}}{\frac{Y}{L}} = \frac{\delta \frac{K}{L}}{\left(\frac{K}{L}\right)^\alpha}$$

or

$$\frac{K}{L} = \left(\frac{\frac{I}{Y}}{\delta_k}\right)^{\frac{1}{1-\alpha}}. \tag{19}$$

With $\theta = 1$, Equation (5) can we written as

$$\frac{c_{a+1}}{c_a}\frac{\beta_a}{\beta_{a+1}} = 1 + r\left(1 - \tau\right). \tag{20}$$

Since $r = r_k - \delta_k = \alpha K^{\alpha-1}L^{1-\alpha} - \delta_k$ we can write this as

$$\frac{c_{a+1}}{c_a}\frac{\beta_a}{\beta_{a+1}} = 1 + \alpha K^{\alpha-1}L^{1-\alpha}\left(1 - \tau\right) - \delta_k\left(1 - \tau\right)$$

or

$$\left(\frac{K}{L}\right)^{1-\alpha} = \frac{\left(1 - \tau\right)\alpha}{\frac{c_{a+1}}{c_a}\frac{\beta_a}{\beta_{a+1}} - 1 + \delta_k\left(1 - \tau\right)}. \tag{21}$$

Equation (19) into (21) yields

$$\frac{\alpha\left(1 - \tau\right)}{\frac{c_{a+1}}{c_a}\frac{\beta_a}{\beta_{a+1}} - 1 + \delta_k\left(1 - \tau\right)} = \frac{\frac{I}{Y}}{\delta_k}.$$

Substituting for $\frac{c_{a+1}}{c_a}\frac{\beta_a}{\beta_{a+1}}$ from Equation (20) into this and solving for δ_k gives

$$\delta_k = \frac{\frac{I}{Y}r}{\alpha - \frac{I}{Y}}. \tag{22}$$

From this we set $\delta_k = 0.065$.

A3.2 Tax Rate, Prices, and Ratios

In this subsection we find the budget clearing tax rate, prices of inputs, and several ratios that will be needed later. The government budget constraint in Equation (10) simplifies to

$$\tau\left(n_w + n_r\right) + \tau_\ell\left(\left(1 - \alpha\right)Y + \sum_{a=0}^{n_w+n_r-1} s_a r\right) = g_e Y + g_u Y$$

and since $\sum_{a=0}^{n_w+n_r-1} s_a = K = \frac{I}{\delta_k}$, this is

$$\tau_\ell = \frac{(1-z)(g_e+g_u)}{1-\alpha+r\frac{1}{\delta_k}\frac{I}{Y}}. \tag{23}$$

$$\frac{\tau}{Y} = \frac{z(g_e+g_u)}{n_w+n_r}. \tag{24}$$

where z is the share of total revenue collected from lump sum taxation. We set $g_e = \frac{G_e}{Y} = 0.0535$ to match education as a share of output in the data described in Section 3. In Section 3, we consider different types of non-education expenditures. For our calibration, we set $g_u = \frac{G_u}{Y} = 0.0204$ corresponding to the share of output spent on public safety. The value of δ_k comes from Equation (22) and other items have already been calibrated. As discussed in the paper, we set $z = 1$ in our baseline calibration but consider other values as well.

We can use Equation (19) to find $\frac{K}{L}$ and from this we find

$$w = (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \tag{25}$$

and

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{\alpha}. \tag{26}$$

From the goods market clearing condition (Equation (12)) we have

$$Y = C + I + G_e + G_u$$

where $C = \sum_{a=0}^{n_w+n_r-1} c_a$ so that

$$\frac{C}{Y} = 1 - \frac{I}{Y} - g_e - g_u. \tag{27}$$

The human capital production function in the steady state can be written as

$$\frac{e}{Y} = B\left(\sum_{j=1}^{n_e}\eta_j\left(g_j g_e\right)^{\varphi}\right)^{\frac{1}{\varphi}}. \tag{28}$$

We set $n_e = 13$ to match the mean years of schooling for workers over the age of 25 in the United States. Recall that g_j is government expenditure on this stage of education so that $G_{e,t} = \sum_{j=1}^{n_e} G_j = \sum_{j=1}^{n_e} g_j Y$. We set each η_j equal to $\frac{1}{13}$ and normalize B to 1. Following Blankenau and Youderian (2015) we set φ=−0.78. From Education at a Glance (2018, p. 256) spending per student in lower secondary and upper secondary education are 8 and 15 percent larger than spending on primary education. With education shares summing to 1, this implies that spending on each of the 9 years of kindergarten through eighth grade is 7.43% of total spending so g_1 through g_9 are set at 0.0743. The proportionally larger values of spending in lower and upper secondary education require $g_{10} = g_{11} = 0.0802$ and $g_{12} = g_{13} = 0.0854$.

A1.3 Vectors of Relative Human Capital, Discount Rates, and Labor Hours

A3.3.1 Relative Human Capital

We next turn to calibrating the sequence of $x_{t,t+a}$ values in Equation (2). Lagos et.al. (2018) show that lifetime wages increase early in a career and peak at between 20 and 30 years of work experience. At this point, they are about 90% higher than during the first four years of work. Since the wage per unit of time is fixed in a steady state, this requires that human capital accumulates through most of life and then falls. This, in turn, requires an increase in human capital sufficient to offset depreciation through most of life. We assume this increase comes simply from gaining more years of work experience. To capture this, we set $\frac{x_a}{h_0} = \Phi_5 \left(a^{\Phi_6} - \Phi_7 a\right) h_1$ so that

$$\frac{h_a}{h_0} = (1 - \delta_e)^a + \Phi_5 \left(a^{\Phi_6} - \Phi_7 a\right). \tag{29}$$

We set δ_e= 0.05 as our benchmark and show results for several other parameter choices. We solve for values of $\mathit{\Phi}_5$, $\mathit{\Phi}_6$, and $\mathit{\Phi}_7$, such that in a steady state equilibrium human capital is maximized at age 43 and is 90% higher at that point than at age 22. Additionally, human capital early in life increases sufficiently to overcome depreciation so that the human capital profile is a smooth, single-peaked function of age.

Specifically we iterate on Φ_6 to satisfy these conditions. For each iteration of $\mathit{\Phi}_6$ we set $\mathit{\Phi}_7 = a^{\Phi_6} a^{\Phi_6 - 1}$ at $a = 24$. Equation (29) is maximized approximately where $a^{\Phi_6} - \Phi_7 a$ is maximized. The expression for $\mathit{\Phi}_7$ comes from setting the first derivative of this equal to zero at $a = 24$. At age 43, $a = 24$ since the agent begins working at age 19 with $a = 0$ and so is 43 years old 24 years later. We assume human capital at $a = 3$ is equal to 1.5 this value at $a = 0$ i.e. $1 = 1.5\left((1-\delta_e)^3 + \Phi_5\left(3^{\Phi_6} - \Phi_7 3\right)\right)$. Knowing $\mathit{\Phi}_6$ and $\mathit{\Phi}_7$, we can solve for $\mathit{\Phi}_5$ using this expression. While human capital may grow rapidly early on, the assumption likely overstates the early pace of accumulation. However it allows us to succinctly provide a reasonable pattern of human capital accumulation from ages 22 to retirement at age 62 as demonstrated in the paper. We iterate on $\mathit{\Phi}_6$ until $\left((1-\delta_e)^{24} + \Phi_5\left(24^{\Phi_6} - \Phi_7 24\right)\right) = 1.9\left((1-\delta_e)^3 + \Phi_5\left(3^{\Phi_6} - \Phi_7 3\right)\right)$. If the created time series is maximized prior to age 24, we increase a by one and repeat. We continue in this way until the created time series is maximized at age 24. This gives $\mathit{\Phi}_5 = -.0137$, $\mathit{\Phi}_6 = 1.73$, and $\mathit{\Phi}_7 = 17.9$. The Bureau of Labor Statistics reports an average retirement age of 62 so we set $n_w = 44$. With these parameters in hand, we use Equation (29) to find the $n_w \times 1$ vector of human capital by age normalized by initial human capital.

A3.3.2 Discount Rates

The steady state counterpart to Equation (5) gives $\frac{c_{a+1}}{c_a} = \frac{\beta_{a+1}}{\beta_a}\left(1 + r\left(1-\tau\right)\right)$. By iteration on this expression we find

$$\frac{c_a}{c_0} = \frac{\beta_a}{\beta_0}\left(1 + r\left(1-\tau\right)\right).^a \tag{30}$$

In the typical case where $\frac{\beta_{a+1}}{\beta_a} = \beta$ so that $\frac{c_a}{c_0} = \frac{(\beta(1+r(1-\tau)))^a}{\beta_0}$, consumption is restricted to grow at constant rate through adulthood. However, consumption over the life cycle is single-peaked. We allow for non-monotonicity in consumption by calibrating non-monotonicity in the discount parameter. Specifically, we normalize $\beta_0 = 1$ and assume

$$\beta_a \left(1 + r\left(1-\tau\right)\right)^a = \left(a+1\right)^{\Phi_1} - \Phi_2 a. \tag{31}$$

We use $(a+1)$ rather than a in the first right hand side expression so that $\beta_a = 1$ with $a = 0$. To match this, we solve numerically for values of Φ_1 and Φ_2 such that in a steady state equilibrium consumption is maximized at c_{31} and $c_{31} = 1.3c_1$. Note that since $a = 0$ at age 19, $a = 1$ at age 20 and $a = 31$ at age 50. Specifically, setting the first derivative of $(a+1)^{\Phi_1} - \Phi_2 a$ with respect to a equal to 0 at age 50 ($a = 31$) gives $\Phi_2 = \Phi_1 32^{\Phi_1 - 1}$. Given this relationship, we choose Φ_1 to minimize $\left(\left(32^{\Phi_1} - \Phi_2 31\right) - 1.3\left(2-\Phi_2\right)\right)^2$. The first expression is $\beta_a r^a$ at age 50 and the second is this value at age 20 scaled by 1.3. This results in $\Phi_1 = .148$ and $\Phi_2 = 0.0077$. According to the Centers for Disease Control and Prevention (CDC), life expectancy in the United States is nearly 79 years so n_r+n_w= $79 - 18 = 61$. We then use Equation (31) to find the 61×1 vector of discount rates given by

$$\beta_a = \frac{\left(a+1\right)^{\Phi_1} - \Phi_2 a}{\left(1 + r\left(1-\tau\right)\right)^a}.$$

A3.3.3 Labor Hours

We set labor hours on average to be one third and choose parameters to match this. Our choice of one third is close to the 31.5% of time spent working found by Somme and Rupert (2007) for individuals aged 16-64. Since we leave out those under 18, who work fewer hours, one third is roughly consistent with this. Like consumption, hours worked are non-monotonic across the life cycle. Because of this, our estimation strategy for γ_a is similar to our strategy for β_a. Specifically, we hold average hours equal to to one third but allow hours worked before normalization in any period, l_a, to vary according to

$$l_a = \left(a+1\right)^{\Phi_3} - \Phi_4\left(a+1\right). \tag{32}$$

The Bureau of Labor Statistics reports that hours worked peak between ages 35-44 and at this point are 65% higher than average hours worked for those aged 20-24. We take the midpoint of these time intervals and calibrate hours worked to match these features of life-cycle labor hours. We solve numerically for values of Φ_3 and Φ_4 such that in a steady state equilibrium hours worked peak at age 40 (with $a = 21$) and are then 65% percent higher than at age 22. Specifically, setting the first derivative of $(a+1)^{\Phi_3} - \Phi_4(a+1)$ with respect to $a+1$ equal to 0 at age 40 gives $\Phi_4 = \Phi_3 22^{\Phi_3 - 1}$. Given this relationship we choose Φ_3 to minimize $((22 - \Phi_4 22) - 1.65(4 - 4\Phi_2))^2$. The first expression is l_a at age 40 and the second is this value at age 22 scaled by 1.65. This results in $\Phi_3 = 0.619$ and $\Phi_4 = 0.188$. We then use Equation (32) to find the 44×1 vector of normalized labor hours

$$\ell_a = \frac{1}{3} \frac{(a+1)^{\Phi_3} - \Phi_4(a+1)}{\sum_{a=0}^{n_w - 1} \left((a+1)^{\Phi_3} - \Phi_4(a+1)\right)}. \tag{33}$$

This assures labor hours peak at the correct time period and at the correct level while equaling one third of available time on average.

A3.4 Levels of Total Labor, Capital, Output and Consumption, and Education Spending

From Equation (13) we have

$$L = h_0 \sum_{a=0}^{n_w - 1} \ell_a \frac{h_a}{h_0}$$

so that from Equation (1)

$$L = e^{\mu} \sum_{a=0}^{n_w - 1} \ell_a \frac{h_a}{h_0}.$$

Thus

$$L = \left(\left(\frac{\frac{e}{Y}}{\frac{L}{Y}\frac{1}{L}}\right)^{\mu}\right) \sum_{a=0}^{n_w - 1} \ell_a \frac{h_a}{h_0}$$

or

$$L = \left(\left(\left(\frac{Y}{L} \frac{e}{Y} \right)^{\mu} \right) \sum_{a=0}^{n_w - 1} \ell_a \frac{h_a}{h_0} \right)^{\frac{1}{1-\mu}}.$$

We set $\mu = 0.85$ as our benchmark and show results for several other parameter choices. We can then find L from Equations (26), (28), (29) and (33).

From this and Equation (19) we find

$$K = \left(\frac{K}{L} \right) L \tag{34}$$

which is sufficient to find

$$Y = K^{\alpha} L^{1-\alpha}. \tag{35}$$

From Equation (27) we have

$$C = \left(\frac{C}{Y} \right) Y \tag{36}$$

and from Equation (28) we have

$$e = \left(\frac{e}{Y} \right) Y. \tag{37}$$

Furthermore, we can find the lump sum tax from Equation (24).

A3.5 Vectors of Consumption, Labor Disutility, Human Capital and Savings

A3.5.1 Consumption

To find c_0 note from Equation (30) that

$$C = \sum_{a=0}^{n_w + n_r - 1} c_a = c_0 \sum_{a=0}^{n_w + n_r - 1} \beta_a \left(1 + r\left(1 - \tau\right)\right)^{a} \tag{38}$$

so

$$c_0 = \frac{C}{\sum_{a=0}^{n_w+n_r-1} \beta_a \left(1 + r\left(1-\tau\right)\right)^a} \tag{39}$$

which can be calculated from Equation (36) and calibrated parameters. Then from Equation (30) and $\beta_0 = 1$ we have

$$c_a = \beta_a \left(1 + r\left(1-\tau\right)\right)^a c_0 \tag{40}$$

yields the 61×1 vector of consumption values.

A3.5.2 Human Capital

From Equations (1), (29), and (37) we find

$$h_a = \left(\left(1-\delta_e\right)^a + \Phi_5 \left(a^{\Phi_6} - \Phi_7 a\right)\right) e^{\mu} \tag{41}$$

which generates the 44×1 vector of human capital.

A3.5.3 Disutility From Labor

We choose the corresponding 44×1 sequence of γ_a values from the steady state analog to Equation (6) ,

$$\gamma_a = \frac{w h_a \left(1-\tau\right)}{c_a \ell_a^{\nu}}, \tag{42}$$

which is calculated from Equations (7), (23), (33), (38) and (41). In doing so we set $\nu = \frac{1}{3}$ so that the Frisch elasticity of labor supply is in the middle of the range commonly used by macroeconomists (Peterman 2016).

A3.5.4 Savings

To find savings we set $s_{a-1}= 0$ with $a = 0$ and $s_a= 0$ when $a = 60$. Together these mean that agents begin and end life with no savings. Otherwise, the 61×1 vector of saving is calculated from Equation (4) such that

$$s_a = w_a h_a \ell_a (1 - \tau) + s_{a-1} (1 + r_a (1 - \tau)) - c_a. \tag{43}$$

A3.6 Capital Market Clearing

Finally, recall that we have, to this point, guessed at the value of $\frac{I}{Y}$. We iterate through this entire calibration until

$$\sum_{a=0}^{n_w+n_r-1} s_a = K$$

using Equations (34) and (43) so that the capital market clears as required by Equation (14). As mentioned above this gives $\frac{I}{Y} = .23$.