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"Bugs" built into the system: How privileged representations influence mathematical reasoning across the lifespan

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Abstract

The initial foundations of human mathematical reasoning appear to be based on "naïve mathematics" — specific and persistent privileged mental representations that develop as a normal part of the human evolved phenotype. Based on the proposed existence of privileged representations in the conceptual domain of mathematics, this paper incorporates findings from early development, childhood mathematical reasoning, and adult statistical decision-making research. The utility of such a framework is demonstrated by analyzing how common errors in fraction and decimal use are explicable in terms of these systematic and reliably developing aspects of human mathematical reasoning. Additionally, the idea that privileged representations continue to exert some influence beyond early childhood holds implications for both research and practice in mathematics education.

Keywords: Cognitive development; Mathematical reasoning; Judgments under uncertainty; Mathematics education; Evolution; Fractions; Decimals

1. Introduction

Although the basic conceptual and procedural knowledge that comprises the academic field of mathematics (rational numbers, fractions, addition, subtraction, etc.) has existed as a relatively discrete and consensually established content area for centuries, the nature and

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cognitive development of that same conceptual and procedural knowledge in the minds of individuals is understood much less well and what is known is of relatively recent vintage. There is a growing consensus that the core foundations of human mathematical reasoning are based on some form of naïve mathematical abilities, even as disagreement continues on the nature, form, and extent of these abilities (e.g., Dehaene, 2001; Geary, 1995; Gelman, 2000). This paper builds on this emerging consensus by outlining how specific elements of this proposed naïve mathematics in cognitive development resonate with recent research in the field of judgments under uncertainty as well as patterns of behavior in the context of learning how to work with fractions and decimals. Drawing these fields of study together leads to suggestions for both research and teaching.

2. Privileged representations in mathematical reasoning

The existence of naïve mathematical abilities relies on the proposal that the development of mathematical knowledge and abilities is guided in significant ways by psychological predispositions (variously called constraints, skeletal principles, intuitive ideas, privileged hypotheses, or privileged representations). Furthermore, these predispositions are domain-specific to certain aspects of mathematics and hence are relatively influential within those areas. In this paper, I will refer to these predispositions as *privileged representations*.[1] Converging lines of evidence supporting the thesis that privileged representations both exist and are necessary come from multiple and independent sources (e.g., Gelman, 1998, 2000; Markman, 1990; Spelke, 1982, 1988, 1990; Tooby & Cosmides, 1992; Wynn, 1995, 1998a, 1998b).

In at least one respect, the success of the privileged representations approach should not be surprising. It splits the difference along one dimension on which previously proposed viewpoints — general learning theory and constructivism — sharply conflicted. *General learning theory* views learning as a relatively passive process in which learners receive information from the environment (e.g., teachers) and incorporate that knowledge into their existing known information (which also came from external tutors). On the other hand, the *constructivist* approach draws more extensively from Piaget and Vygotsky and assumes that children are active learners in the construction of knowledge. That is, the development of knowledge and abilities is constructed by both the environment (particularly the social environment of teachers and peers) and the active processes of the individual. If anything, the student takes the primary role in learning, with the teacher playing a supporting role by

[1] A privileged representation is defined here as a form of knowledge that is reliably developing in normal humans, within the normal range of environments (for example, the grammar structure of language is often argued to be of this form). As will be covered later in this paper, privileged representations are considered here to be derived primarily from the human cognitive architecture specified by evolutionary selection pressures (variously called cognitive adaptations, Darwinian algorithms, and modules), but the term privileged representations is preferred here because it focuses on the nature of the result of these cognitive structures. “Privileged representations” is also roughly analogous to Geary’s term of “biologically primary abilities,” as also discussed later (see also Williams, 1966 classic definition of an evolved adaptation and the more recent definition of Tooby and Cosmides, 1995).

providing appropriate materials and social context (see, for example, Empson, 1999; Tzur, 1999). The notion of privileged representations places learners squarely in the arena of actively contributing to their own knowledge development. At the same time, the external environment is absolutely essential as the primary source of new information to be learned in order to move the learner beyond the basic features provided by these skeletal representations.

In a way, it is unfortunate that there is a labeled "constructivist" viewpoint, as all three of the above viewpoints can be said to involve the construction of mathematical understanding in children's minds. The key issue is the identity of the builders. Although both recognize other contributions, general learning theory focuses on the environment as the builder and the constructivist view emphasizes the learner as the principle builder. The privileged representations view identifies the evolved dispositional structure of the human mind as the initial builder, interacting with the environment in ways that become increasingly important as mathematical reasoning becomes more advanced. Thus, the privileged representations view is in some ways very similar to the constructivist viewpoint as it argues for an active role of the individual (that is, the structure of the individual's mind) but also similar to general learning views in recognizing the importance of the environment. The privileged representations approach, however, does more than strike the middle ground regarding the architects of knowledge. It is also quite different from these other views in certain key respects. First, the privileged representations approach aligns much more closely with the biological ideas of genotype and environment interacting to produce a phenotype or, more simply, the interacting contributions of nature and nurture. Second, the privileged representations approach rejects the assumption of equipotentiality of conceptual structures.

The notion of privileged representations conflicts with a background assumption of equipotentiality to which both general learning and constructivist theories generally adhere. Equipotentiality means that no concepts (in this case, mathematical concepts) are a priori considered to be easier or harder to acquire. To some extent, most researchers recognize that this assumption cannot hold completely: counting whole numbers is certainly easier to acquire than trigonometry, but when instead of dealing with extremes one looks at conceptions within a particular domain (such as dealing with fractions), the equipotentiality assumption is often made. So, for instance, it is assumed that it is equally likely that students will adopt the idea of a fraction being a ratio, a rational number, a division operation, or an alternative expression for decimal numbers. This position rests on a long tradition of minimal assumptions of human nature, "instincts," or reliably developing abilities (e.g., Hume, 1955/1748). Learning is commonly assumed to be based on de novo constructions or expansions and reconstructions of ideas and concepts that were originally de novo.

Differing conceptions of particular mathematical contents can be seen via the several systematic errors that children often exhibit in the course of learning mathematical topics. These errors have been called "bugs" or "buggy algorithms" to indicate that the programming in the child's mind (i.e., learning) contains one or more errors (e.g., Brown & Burton, 1978; Silver, 1986). More implicitly, these buggy algorithms are assumed to reflect conceptual or procedural knowledge that was recently created for some novel purpose (this is an aspect of the computer programming metaphor from which the term "buggy" is taken). Remediation of these errors requires additional learning (i.e., reprogramming of the cognitive algorithms to eliminate the errors), and the ease with which bugs can be fixed is assumed to be a direct function of how well the buggy algorithm manages to perform, despite being erroneous.

From both a theoretical and an empirical standpoint, the equipotentiality assumption has problems. In principle, a system that starts with all possible representations as equally probable must use some relatively abstract (i.e., content independent and general purpose) process to generate a preference structure of representations. An example of such a mechanism is one that takes in examples and definitions provided by the environment, notices commonalities, and induces general concepts or procedures from these groupings of experience. Repeatedly, however, it has been found that such open-ended mechanisms are unable to solve the problems to which they are applied, and the reason for these failures appears to be inherent to these types of systems. The problem, essentially, is that given a specific event (or group of events), there is an infinite number of compatible explanations (i.e., possible general concepts or production procedures). General problem solvers only work for totally general problems and the real word almost never involves completely general, abstract problems. This basic phenomenon can be seen throughout the sciences: in linguistics where Chomsky (1975) labeled it "the poverty of the stimuli," in statistics where it is manifested as combinatorial explosion, in philosophy where it is referred to as the problem of induction (or even the "scandal of philosophy"), and in artificial intelligence where it is referred to as the "frame problem." It has also been (albeit less directly) a force for understanding how the human visual system works (Shepard, 1984, 1992; see also Simon, 1973 on ill-structured versus well-structured problems). The very divisions into which psychology is traditionally divided implicitly recognize the incompatibility — and hence specificity — of different cognitive tasks. Perception is different from language, which is different from memory, which is different from social behavior, which are all different from mathematical reasoning.

Research findings demonstrating that the equipotentiality assumption is generally problematic go back several decades now (Garcia & Koelling, 1966). Several research findings over the last decade have documented aspects of cognitive development that appear to violate the assumption of equipotentiality within the domain of mathematics learning. It is now fairly well established that even very young children come prepared to learn certain, specific properties of numbers, and these differentially prepared states evidently continue through adulthood. Gelman (1998, 2000) and Wynn (1992a) summarize much of the experimental evidence regarding the acquisition of numerical knowledge. Studies across species, with infants, with children, and with adults, all support the basic contention that some basic mathematical concepts (as well as cognitive abilities more generally) develop with the required aid of specific privileged representations. There is still controversy as to the exact nature of these privileged representations, but as a basic general statement, one can fairly and safely say that some simple mathematical abilities are part of a universal human nature and not arbitrary cultural constructions. Furthermore, many believe that these abilities form an "intuitive mathematics" (or naïve mathematics, or skeletal principles, or biologically primary abilities, etc.) that is a reliably developing aspect of the normal human phenotype (e.g., Geary, 1995; Geary & Lin, 1998; Gelman, 1998, 2000; Wynn, 1992a, 1992b, 1995, 1998a, 1998b). Most previous work in this tradition has focused on relatively fundamental numerical knowledge, such as numerosity, ordinality, and whole number addition in infants and very young children. Little work in developmental psychology has gone into if and how these privileged representations that predispose young children to perceive numerical information in certain ways influence more advanced mathematical abilities.

3. Privileged representations in judgments under uncertainty

The role of privileged representations has, however, been explored in the context of advanced mathematical reasoning within the field of judgments under uncertainty. Like the "buggy algorithms" view of children's learning, many researchers had been led to believe that sophisticated mathematical tasks such as inferring posterior probabilities are governed by adequate yet ultimately flawed "heuristics and biases" (Kahneman, Slovic, & Tversky, 1982). This viewpoint has been challenged by a number of researchers taking a theoretical position that has recently been called *ecological rationality* (Gigerenzer, Todd, & The ABC Research Group, 1999). Ecological rationality proposes that the mind is designed to function in ways that reflect and utilize the regular properties of the world (more precisely, the properties of the worlds in which our ancestors' minds evolved). The aspect of ecological rationality that is particularly of interest here is how the mind views the world and imputes structures (as opposed to the aspects of the world that the mind simply takes advantage of in making decisions). A growing body of research indicates that there are privileged representations that adults use in making judgments under uncertainty (Brase, Cosmides, & Tooby, 1998; Cosmides & Tooby, 1996; Gigerenzer, 1991; Gigerenzer & Hoffrage, 1995; Gigerenzer et al., 1999; Hoffrage, Lindsey, Hertwig, & Gigerenzer, 2000). Some of these privileged representations are:

Frequencies, as opposed to formats such as single event probabilities (0.05) or fractions (1/2), are a privileged representational form (cf. Johnson-Laird, Legrenzi, Girotto, Legrenzi, & Caverni, 1999, but see related comment by Brase, in press). Obviously, this ties in directly with the findings in developmental psychology on the naïve mathematics of the natural numbers, but beyond that, it has been pointed out that information about the occurrences of objects and events in the world would most reasonably yield frequency counts.

Natural sampling is a system of tabulating frequency counts within nested (set/subset) categories. Thus, for example, one could have counts of how many apple trees you have seen, how many orange trees you have seen, and the combined total of how many fruit trees you have seen. This would, again, provide a close correspondence between the cognitive system and the nature of the environmental information being used by that system (and it also has some implications for procedures such as inferring posterior probabilities; see Gigerenzer & Hoffrage, 1995).

The individuation theory qualifies some of the work with frequencies and natural sampling by pointing out that a system that tracks frequencies must have a priori rules for what is counted and what is not counted. The basic rule proposed by this theory is that whole objects, events, and locations are readily encoded as frequency counts, whereas aspects or properties of those items are difficult to track (Brase et al., 1998; see also the developmental work of

Shipley & Shepperson, 1990; Spelke, 1988, 1990, which formed one line of evidence used in developing this theory).

Pictorial representations generally seem to improve mathematical reasoning as well. It has been pointed out (Brase et al., 1998; Cosmides & Tooby, 1996) that most information fed into these natural sampling systems (during their evolution as a part of the human cognitive architecture) would have been visually perceived real objects, real events, and real locations. The fact that these systems work at all using written word problems with Arabic numerals instead of counts based on actual experience is a testament to the flexibility of the human mind. Nevertheless, it should be expected that better approximations of real stimuli, such as pictorial representations or token objects, will improve subsequent mathematical reasoning (e.g., see Mix, Levine, & Huttenlocher, 1999).

In a watershed article, Geary (1995) focused on differences between biologically primary abilities (those that have heavy influences of privileged representations) and biologically secondary abilities (with little or no influences of privileged representations). The present paper agrees with this overall view but takes a somewhat different approach. Whereas some might consider the biologically secondary abilities as, therefore, "free" of privileged representations, guiding constraints, or skeletal principles, this paper seeks to provide a specific illustration of how certain biologically secondary abilities — fraction and decimal understanding — can be influenced by privileged representations. This implies, for example, that people untutored or still developing knowledge in more complex (i.e., biologically secondary) mathematical procedures and concepts, such as fractions and decimals, will tend to interpret numbers as frequency counts. In other words, learning of secondary abilities rests on the foundation of biologically primary abilities. Furthermore, this biologically primary frequentist interpretation on the part of learners can be expected to be relatively resistant to simple modification or elimination. Thus, certain characteristic behaviors can be expected to commonly emerge when relatively naïve mathematics students are presented with math problems. Although one might simplistically write off these assumptions as "bugs," they may very well instead be indicative of sophisticated algorithms that were designed by natural selection to reliably develop in humans.[2]

It may be useful to consider an increasingly common euphemism in the computer software industry: The description of computer software problems (computer bugs) as "undocumented features" of the program. The ploy is that the unanticipated behavior of the software program is a purposefully designed part of the program. In reality, undocumented features of computer programs are usually simply unforeseen programming glitches (i.e., bugs) and the name change truly is a deceptive ploy. However, the human mind was not programmed by Microsoft™ but was programmed by natural selection over millions of years and with continual feedback from extensive field testing. "Bugs," in the direct analogy sense, does not hold as well as it might initially appear. Instead, the mind is filled with true "undocumented features"

[2] This does not mean that these "buggy" responses should now be considered correct answers. The field of mathematics, independent of considerations of human cognitive development, continues to apply in matters of determining correct answers to mathematics problems.

(thus, experimental psychologists have jobs). To put it another way, if my word processor puts a red line under the word "frequentist" (as it does) and I do not realize that this is a spell-checking function, I may conclude that the computer program is bugging up my paper and I would be labeling an undocumented feature as a "bug." If, however, I understand this as a spell-checking function, I can figure out how to turn it off when I do not want it putting red underlines throughout my documents. Understanding why a certain response occurs can be key to understanding how to either get that response or circumvent the occurrence of that response.

4. Privileged representations and working with fractions and decimals

Fractions and decimals are related constructs. They both can be defined as ways of expressing a division of whole numbers. Fractions accomplish this by giving a portion of some given unit of equivalent division (e.g., 3/20 is 3 parts of what has been divided into 20 equal parts). Decimals give a portion of base-10 equivalent divisions (e.g., 0.15 is 1 part of what has been divided into 10 equal parts and 5 parts of what has been divided into 100 equal parts). The difference between them is, therefore, primarily whether the unit of equivalent division is set to some standard (base-10) or allowed to vary according to the properties of the situation. Decimals, in fact, can be more formally called "decimal fractions." Fractions and decimals both are methods for moving beyond the natural numbers and expressing more finely graded quantities (i.e., rational numbers).

4.1. Fractions

The fact that elementary school children have difficulty understanding and using fractions is something generally agreed upon by both researchers and teachers alike. Gallistel and Gelman (1992, p. 69), for example, call the teaching of fractions "a major pedagogical challenge." There are, of course, many opportunities for difficulties in children's mathematical education prior to the introduction of fractions, but acquiring the conceptual and procedural understanding of fractions appears to be a major area that is particularly prone to trouble. Specifically, problems with fractions manifest themselves not only as straightforward difficulties in acquiring a basic and accurate conceptual understanding of fractions but also in the acquisition of faulty procedural understandings of working with fractions. For example, a common error in learning to add fractions is to simply add the numerators and add the denominators to reach an answer (e.g., $1/2+1/3=2/5$). The problem, as Mark Twain put it, is "not what people do not know but what they know that just is not so."

The example just given is common enough that it has been dubbed the "freshman bug" or more descriptively the "rational number addition bug." The common explanation is that these procedural errors are diagnostic symptoms of underlying flawed or incompletely integrated conceptual knowledge. According to Silver (1986, p. 191), "it is fairly common and reasonable to attribute the error, when it is made by young children, to an incorrect generalization of whole number addition." This is a perceptive account of what a rational number addition bug is, but it really does not address the origin of the bug or why it occurs in such a large number of children. To understand the origins of the rational addition bug, we must first understand something about the nature of fractions.

Consider the current example of the tendency for students to answer a problem like "1/2 + 1/3" with "2/5" (i.e., [1 + 1]/[2 + 3]). This response should be an expected, albeit erroneous, response if one understands that frequency representations are privileged in the human mind. If people are predisposed to view numbers as frequency counts, "1/2" and "1/3" are plausibly treated as ratios of frequencies (1 out of 2 and 1 out of 3) rather than as normalized fractions (1 ÷ 2 and 1 ÷ 3). The perfectly correct answer, given such a frequency ratio interpretation, is, in fact, "2/5." The rational number addition bug is a manifestation of an undocumented design feature (frequencies as a privileged representational format). To obtain the answer that is usually desired ("5/6"), certain properties must be assumed to consider the structure of one-number-over-another as properly being a normalized fraction. These properties are:

(A) *Base unit property*: The denominator (the base unit) for all the fractions must be:
   1. From the same unit, or
   2. If not from the same unit, they must
      (a) be units of the same size,
      (b) combining only the sampled segments, and
      (c) evaluated based on the original base unit size.

(B) *Unique segments property*: The numerator (the sampled segment) for all the fractions must be nonoverlapping entities.

Given that the assumption of these properties is clearly met, then one can correctly proceed with the normative mechanics of adding the fractions together. Rittle-Johnson and Siegle (1998, pp. 96–97) recognized aspects of these properties when they noted that: "they [students] must learn that the two numbers within a fraction represent a single quantity, that units equal in size are necessary for adding and subtracting fractions, and that the same amount can be represented with fractions that include different numbers (e.g., 1/2 = 3/6)." Table 1 shows some simple examples of mathematics word problems that vary in the satisfaction of the above properties. Note that one cannot violate all these properties simultaneously (e.g., One cannot have sampling from different units and also overlapping samples or have sampling from different sized units and maintain an "original" base unit size). On the other hand, it is possible to violate several different properties within a single mathematics problem.

By various stretches of meaning, however, all the problems in Table 1 can be notated as "1/2 + 1/3=?" Yet, the content of the word problems clearly makes them nonequivalent in both their difficulty and their solutions. It is clear that there is not just one way of combining fractions, and while the normative answers of the classroom are certainly the most generalizable conclusions, other interpretations may be more than simply "wrong." Some children may see "1/2 + 1/3" and interpret it as something like "one brown egg out of two eggs is combined with one brown egg out of three eggs," for which the correct answer actually is "3/5." In such a case, this is not a simple bug to be reprogrammed just as easily as it was

Table 1
Various ways in which the assumptions underlying fraction addition can be violated, with attendant examples

Violations	Example problem
No violations (correct answer: 5/6)	You have a carton of eggs that you bought from a local farm. 1/2 of the eggs in this carton have white shells. 1/3 of the eggs in this carton have brown shells. The remaining eggs are gone (used for cooking). What fraction of the carton of eggs is left?
Base Unit 1 violated, but Base Unit 2 intact (correct answer: 5/6)	You have two cartons of eggs that you bought from a local farm. 1/2 of the eggs in the first carton have brown shells. 1/3 of the eggs in the second carton have brown shells. The remaining eggs are white. You take the brown eggs out of both their original cartons and place them together in a new (empty) carton. Then, you fill in the rest of the new carton with white eggs. What fraction of the new carton is filled with brown eggs?
Unique Segments Assumption violated (correct answer: cannot say)	You have a carton of eggs that you bought from a local farm. 1/2 of the eggs in this carton have brown shells. 1/3 of the eggs in this carton have cracked shells. The remaining eggs are gone (used for cooking). What fraction of the carton of eggs is left?
Base Units 2a and 2c violated (correct answer: cannot say)	You have two packages of eggs that you bought from a local farm. The first package of eggs came in a large basket, and 1/2 of the eggs in the basket have brown shells. The second package of eggs came in a carton, and 1/3 of the eggs in the carton have brown shells. The remaining eggs are white. You take the brown eggs out of both their original containers and place them together in a new (empty) box. What fraction of the new box is filled with brown eggs?
Base Units 2b and 2c violated (correct answer: cannot say)	You have two cartons of eggs that you bought from a local farm. 1/2 of the eggs in the first carton have brown shells. 1/3 of the eggs in the second carton have brown shells. The remaining eggs are white. You take all the eggs out of both their original cartons and place them together in a basket. What fraction of the basket is filled with brown eggs?
Base Unit 2c violated (correct answer: 1)	You have two cartons of eggs that you bought from a local farm. 1/2 of the eggs in the first carton have brown shells. 1/3 of the eggs in the second carton have brown shells. The remaining eggs are white. You take the brown eggs out of both their original cartons and place them together in a basket. What fraction of the basket is filled with brown eggs?
Base Units 2a–2c violated (correct answer: cannot say)	You have two packages of eggs that you bought from a local farm. The first package of eggs came in a small basket, and 1/2 of the eggs in the basket have brown shells. The second package of eggs came in a large carton, and 1/3 of the eggs in the carton have brown shells. The remaining eggs are white. You take all the eggs out of both their original containers and place them together in a new (empty) box. What fraction of the eggs in the new box are brown eggs?

putatively written in by the child's earlier experiences. This can instead be an example of a child not making certain property assumptions because those properties have not been explained or justified in any way. One can think of this process as a "default setting" in the mind of a child as to how to consider numbers. Such a starting point would often be useful in terms of providing some computationally necessary problem structure. As Clements and Del Campo (1990, p. 186) pointed out, "At various times of their schooling, children are told that the fraction 1/3, for instance, is concerned with each and all of the following: (a) sharing a continuous quantity between three people, (b) sharing 12 (say) discrete objects between three people, (c) dividing the number 1 by the number 3, (d) a ratio of quantities, (e) a 1 for 3 replacement operator, (f) a rational number equal to 2/6, 3/9, and so forth, and (g) a decimal fraction of 0.333..."

4.2. Decimals

Learning to work with decimals is another notorious area in mathematics education. Decimals are typically taught after students have learned fractions, and by many accounts, "the decimal number system comprises the lion's share of the elementary and junior high school mathematics curriculum" (Hiebert & Wearne, 1986, p. 200). This sequence is sensible, as Hiebert, Wearne, and Taber (1991, p. 322) point out, because "Decimals are rather complex mathematical entities. They represent a confluence of common fractions and whole numbers. Stated most simply, decimals use base-10 (whole number-like) notation to stand for fractional quantities. This simple-sounding statement carries significant meaning. Fractional quantities do not necessarily measure a whole number of units. They can have a continuous nature, such as length or weight."

With this greater complexity comes a larger variety of documented "bugs." A well-known example of a decimal-based buggy algorithm is the phenomenon labeled "Benny's bug" (Erlwanger, 1973), which involves decimal math problems such as "$0.2+4.0=?$" Initially, many students consistently produce the answer "0.6" to this problem. While this is not the correct answer, it seems to be much less of an arbitrarily wrong answer in light of recent work on the primacy of frequency representations in statistical reasoning. What produces this bug is the dominance of a whole number (i.e., frequentist) perspective of numbers. Other decimal-related bugs include the following:

(a) Adopting the rule that "more digits means bigger" (e.g., 0.1234 is larger than 0.32)
(b) Adopting the rule that "more digits means smaller" as a reaction to learning that the first rule is wrong (e.g., 0.4321 is smaller than 0.23)
(c) Adopting a rule that attaching zeros to the right of decimal number increases the size of that number. (e.g., $0.8<0.80<0.800$)
(d) Adopting a rule of ignoring zeros on the left (e.g., $0.8=0.08=0.008$)
(e) Adopting a rule of ignoring decimals, thus lining up digits on the right rather than lining up the decimal points (e.g., Benny's bug: $0.2+4=0.6$, and variants: $0.07+0.4=0.11$, $6\times 0.4=24$, and $42\div 0.6=7$)

All of these responses are explainable both methodologically and ontogenetically by understanding that frequentist representations of numbers tend to dominate the intuitive psychology of mathematics. In fact, the common patterns of these "bugs" have not escaped researchers in this area. Hiebert and Wearne (1986, pp. 204–205) note that "Extending concepts of whole numbers into referents that are appropriate for decimal fraction symbols is a delicate process. Students must recognize the features of whole numbers that are similar to decimal fractions and those that are unique to whole numbers. . .The evidence suggests that many students have trouble selecting the features of whole number that can be generalized. . .Most errors can be accounted for by assuming the student ignores the decimal point and treat the numbers as whole numbers." What has again been more elusive in these accounts, and is provided here, is a way to understand *why* this aspect of learning about decimals is so commonly problematic and an explanation that fits these phenomena into a larger picture of mathematical development and education. Earlier explanations have only proposed that these buggy procedures result from overgeneralizations of a familiar mathematics domain (whole numbers) to a new domain (decimals/fractions; e.g., Resnick et al., 1989). Why, then, do whole number bugs persist in the learning of decimals when (under the equipotentiality assumption) fractions could be just as easily used as the familiar domain from which decimals could be generalized? The answer may be that the use of decimals involves a different set of background properties that must be met in order to overcome the representation of numbers as being simple frequencies (see Resnick et al., 1989 for a similar comparison). These background properties include:

(A) Using the decimal point as a key reference point that determines the status of the numbers before and after it
(B) Assigning meaning to numbers based not only on their face value but also based on their positions, relative to the decimal point
(C) Using zeros as "placeholders" rather than contributing to the value of the number

Comparing these property assumptions to those implied in the use of fractions, one can see that decimals actually involve less background to understand when starting from a default representation of whole numbers. This would help explain why Moss and Case (1999) found that introducing decimals before fractions actually led to better pedagogical outcomes than the more traditional sequence of teaching fractions before decimals.

5. Implications for research

What is the utility of holding frequencies as a privileged representational format? That is, how does this improve upon the idea that buggy algorithms are overgeneralizations from a familiar domain (whole numbers) to a novel domain (fractions or decimals)? First, it provides a stronger theoretical grounding for the very fact that whole numbers are invariably the "familiar domain" and decimals and fractions are invariably the "novel" domains. As intuitive as this situation seems, it is a property of how the human mind constructs (and in turn has constructed) the world, and it is a violation of equipotentiality. Second, this viewpoint usefully integrates research with adults (in judgments under uncertainty), with children (in mathematical reasoning), and with infants (in perception of numerosity). The argument that fractions and decimals are "novel" domains does not work so well when explaining the better performances of university students using frequencies (i.e., whole numbers) as compared to other numerical formats. Furthermore, this integration of research findings illustrates how the naïve mathematics of childhood are not eventually subsumed by years of teaching but continue to exert their influence well into adulthood. Finally, the merits and existence of domain-specific, content-dependent mechanisms (e.g., privileged representations) in juxtaposition to more general-purpose, content-independent mechanisms (e.g., general learning) is a broader debate within psychology that has been explored elsewhere at some length (e.g., Tooby & Cosmides, 1992). These alternative viewpoints point to very different conceptualizations of the human mind generally.

5.1. On the nature of mental representations of frequency

There is genuine debate as to the form of magnitude representations in the mind. That is, even if information enters the cognitive system as frequencies, does it retain that form (i.e., digital) indefinitely or does the magnitude information at some point become an analog representation? There is good evidence that at least some magnitude information is accessed in terms of analog representations (Huntley-Fenner & Cannon, 2000), and the proposal that the human cognitive system is designed to work with frequency information is actually untroubled by the fact that some outputs of the system appear to be analog. After all, it is not necessary (perhaps not even desirable) that someone be able to say that they found food in the East valley 234 out of 300 times and found food in the West valley 97 out of 200 times. What is important is that that person be inclined to preferentially go to the East valley when hungry (see also Klein, Cosmides, Tooby, & Chance, in press).

Once again, it is instructive to consider the environment in which the human mind evolved. The proposition that people track the frequencies of objects and events within a natural sampling system does not require that those people have open access to the actual frequency counts. In fact, it would be likely over most of human evolutionary history and ecological circumstances that people did not have the symbolic language to adequately express many of these natural sampling results. Conversely, as noted earlier, the use of symbolic notations for magnitude (i.e., written numbers) is less effective in producing accurate mathematical reasoning than counting actual objects or events. Interestingly, Mix et al. (1999) have also found some fascinating results of early competencies in using pictorial analogs of fractional quantities.

5.2. On the relationship between conceptual and procedural knowledge

A large body of literature exists regarding the relationship between conceptual knowledge and procedural knowledge in mathematics education. Clearly, the notion of privileged representations proposes a situation that can be understood as certain forms of conceptual knowledge preceding procedural knowledge. It is important to emphasize, however, an aspect in this conceptual/procedural distinction that is sometimes overlooked: the existence of conceptual knowledge implied to exist by the manifestation of procedural competence does not necessarily imply conscious accessibility of that conceptual knowledge. A simple example of this is when a child throws a ball — something that involves the physics of force, momentum, and gravity. The child *consciously* accesses none of these concepts, but all these concepts are implicitly used in the successful throwing of a ball (see also a similar example, using language, in Gelman, 2000).

The principles embodied by privileged numerical representations are not necessarily or automatically open to conscious access, any more than the laws of physics, optics, acoustics, or grammar are consciously accessed in walking, seeing, hearing, or speaking. For this reason, children (and adults) can enact procedures for which they do not "know" the rationale (i.e., there is no conscious access to the underlying conceptual principles). One can view a part of the educational process as not only teaching procedural skills but also teaching conceptual information that may or may not be implicitly represented in other parts of the mind (e.g., Rozin, 1976).[3]

Some research with adults, along the methodological lines of judgment under uncertainty research, may be useful in empirically establishing how and to what extent the background assumptions of fraction and decimal mathematics are employed (e.g., by giving the problems in Table 1 to participants), but it would perhaps be more interesting and more useful to move towards applied research with children. Two possibilities for research of this type are immediately suggested: (1) comparisons of teaching methods using matched classrooms, in which one group of students experience standard teaching methods regarding a topic (e.g., adding fractions) and the other group of students experience teaching methods that based more strongly on the principles outlined in Section 6, and (2) research using remedial education efforts that are based on the principles outlined in Section 6 (again compared to some control group), perhaps similar to the research conducted by Rozin (1976) on developing reading skills. Finally, there may be some useful implications of this approach, for example, looking at the individual assumptions for fraction use, in relation to research on mathematical abilities of nonhuman species (e.g., Beran, Rumbaugh, & Savage-Rumbaugh, 1998; Boysen & Hallberg, 2000; Brannon & Terrace, 2000; Davis & Perusse, 1988).

6. Implications for teaching

This paper now turns to the implications of the above considerations for the teaching of mathematics. Many of these implications are not entirely, or even primarily, novel. Mathematics teachers have spent hundreds of years noting which topics are more difficult, which learning aids are helpful, and which teaching techniques work. In discussing these implications for teaching, it is hoped that, by developing a deeper understanding of why the aides and

[3] This is also not a purely socratic view that all knowledge is pre-existing in the mind and needs to simply be drawn out. The fact that one cognitive system uses some phenomenon that has been adequately labeled and described by science does not make the conceptual understanding of that phenomenon intrinsically easier. The most that can be claimed generally is that a cognitive system that manifests a particular conceptual principle can make for a particularly useful example to help someone explicitly understand that same concept.

obstacles in learning mathematics are structured as they are, effective learning methods can be applied more wisely and possibly even extended in effectiveness (Geary, 1995 also makes several excellent suggestions regarding pedagogy that fit with this approach).

6.1. Anticipate the nonequipotential nature of concept learning

That is, plan for a significant number of learners initially understanding fractions as subsets of whole objects. It is important to realize that this pattern of learning is not necessarily the result of "bad" teaching and that children are not learning mathematics in a vacuum, with only their classroom instruction providing information. The evolutionary history of humans has programmed dispositions into the minds of all normally developing individuals, such that certain numerical representations will be privileged over other representational formats. Rather than railing against these privileged representations with rote memorization and noninsightful procedural rules, one can use an improved knowledge of where children are "coming from" to guide them in the directions that mathematics instruction needs to take them.

6.2. Include explicit discussions of deviations from privileged ideas

Good teachers do not just know what they want to teach but they also know what the learners already understand. In other words, teachers should not make the equipotentiality assumption but instead actively point out and reveal the all too often implicit properties that underlie the use of fractions and decimals (e.g., as outlined in this paper; see also Resnick et al., 1989).

6.3. Include instructions on "debugging"

This is in extension to the previous point. When privileged representations make correct mathematical reasoning more difficult, learners should be specifically taught how to reason about mathematics correctly. When "buggy" reasoning does occur, more needs to be done than simply telling the student they are wrong and how to work out the correct answer. Our understanding of how the mind represents numbers allows us the opportunity to explain *why* the student's initial response is wrong and at the same time point out the underlying reasonableness for making such an error.

6.4. Value both procedural and conceptual knowledge

There is always a tension between teaching procedures — which are easy to apply and get immediate results but prone to misapplications — and teaching concepts — which are harder to teach, yield more distant results, but are longer lasting and often more accurate in application. Rather than pass a blanket judgment on which type of knowledge is preferable, the privileged representations view indicates that the merits of teaching procedural or conceptual knowledge change with the topic at hand. A mathematical concept that meshes well with the intuitions of naïve mathematics can be taught first with little trouble, with procedural knowledge following the concept acquisition. Other concepts are more difficult to acquire because they run counter to the privileged representations within naïve mathematics. These concepts may be better left until after some procedural knowledge has been established to reinforce the relatively difficult concept acquisition. There is a danger here that some learners will not progress beyond the procedural knowledge in this later situation, and this possibility should be guarded against (Tirosh, 2000).

6.5. Frame learning situations

Although it would appear that there is little one can do to alter the nature of the privileged representations that form the basis of the human naïve mathematics, it would be grossly inaccurate to say that there is little one can, therefore, do to change how concepts are represented in the process of learning about mathematics. One key to developing desired representations from the start is to frame the learning situations to promote desired representations. Note that this is similar to some ideas within the constructivist viewpoint but notably different in that is advocates a much more directive role for the teacher. Several research programs, working from various premises, support this approach. Miura, Okamoto, Vlahovic-Stetic, Kim, and Han (1999) argued that Korean students performed better in understanding the part–whole quantitative relationship that exists in fractions because the concept of that relationship is embedded within the mathematics terminology of East Asian languages. Moss and Case (1999) found that an inversion of the typical order of introduction for fractions and decimals (i.e., they introduced decimals first and then fractions as an alternative notation style) led to better performance on measures of both conceptual and procedural knowledge. Finally, in his work with teaching children fractions, Silver (1986) noted in passing that remediation of buggy algorithms with fraction bars and cardboard regions did not work well, but using measuring cups worked much better. This finding was not pursued further by Silver because "It is not clear what aspects of the measuring cup image were most helpful to subjects. It is possible that other alternative models would also be effective" (p. 196). From a perspective that includes the existence of privileged representations, it is easy to recognize that the crucial aspect in these different mediums for remediation is the individuation of items within the materials (i.e., discontinuous objects that can be clearly counted as frequencies). Items that can be divided into discrete, countable units (bars, pieces of cardboard, slices of pizza, etc.) will tend to invoke frequentist representations of multiple items (Brase et al., 1998; see also Sophian & Kailihiwa, 1998). On the other hand, items that are perceptually continuous (e.g., liquids, sand, sugar, and flour—things that typically use measuring cups) will not be easily represented as multiple, discrete, and countable units (more abstract continuous items, such as time, will quite likely work in a similar manner). From a pedagogical standpoint, it would probably be most productive to demonstrate concepts such as fractions with both forms of representation and then discuss the similarities and differences between the media used and the mathematical procedures involved.

7. Further issues

This paper has dealt with some common "buggy algorithms" and attempted to show how a model of privileged representations that includes intuitive assumptions about the existence and nature of frequentist numbers in the world can help us to understand these phenomena. The recognition of privileged representations in the mind may not be relevant to every type of mathematics "bug" that exists, and it is important to make this limitation clear. There are, however, also privileged representations other than those considered here that just as certainly might influence the development of mathematics knowledge and skills. It can be informative, therefore, to review a few other findings in the literature that seem to fit within the explanatory domain of privileged representations being assigned as "buggy."

Hartnett and Gelman (1998) took a perspective similar to the one developed in much more detail here: that "inputs about fraction may not be interpreted as intended by the school but rather in terms of the child's theory that number is *what one gets when one counts things*...Doing this amounts to treating the data as if it were made up of novel counting examples as opposed to exemplars of a new kind of number" (p. 363). They found that whereas children readily learned a concept such as infinity, which is consistent with frequentist representations of numbers, children had great difficulty in rank-ordering numbers with fractional notations.

The findings of Sophian and colleagues also fit with the idea of a frequentist representation that tends to parse the world into discrete, countable units (Brase et al., 1998). Sophian and Wood (1997) found that children usually begin with part–part relationships when dealing with proportions (e.g., comparing different parts to each other) and then later develop an ability to use part–whole relationships (e.g., comparing a part of an object to the whole object). Sophian and Kailihiwa (1998) found that when children counted arrays of items, "It was possible to identify some common unit — whether whole object, pieces, or discrete things — in virtually all the counts the children produced" (p. 583). Sophian (2000) found that young children (under 5) can take account of object quantities and object sizes but have difficulty in aggregating across objects to determine combined size.

Finally, there may be alternative explanations for the privileged representational status of frequencies. Most notably, one could argue that fractions and decimals (and other alternative formats) are inherently more complex than frequencies. Therefore, frequencies are of course learned more easily, more quickly, and more pervasive. An analogy would be that it is simply more difficult to walk up a steep hill than to walk along flat ground. The question, in brief, is whether a particular format is a privileged representation by virtue of evolved dispositions that make it so or by virtue of just happening to be inherently simpler to begin with. The crux of distinguishing between these alternatives actually depends on the existence (or lack of) of an external, environmental reference factor. That is, the analogy of walking up a hill being harder than walking on level ground depends crucially upon the existence of gravity — a factor wholly outside of considerations of psychology and experiences such as "difficult" and "easy." To establish that the ultimate explanation for frequencies being a privileged representational format is just their inherent simplicity compared to other formats, one must identify an external, environmental reference factor that makes this distinction separate from human judgments. In other words, frequencies may just seem less complex than other numbers because our minds are designed to specifically to work with frequencies.

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