Lab 9: Projection Matrices in Excel

A range of software tools can be used to calculate the properties of projection matrices, including programming languages such as SAS or specialized math software like Matlab. In this exercise, we will use the 'power method' in Excel to calculate the dominant eigenvalue (λ or the rate of population change), the right eigenvector (**w** or the stable age/stage distribution) and the left eigenvector (**v** or the reproductive value). The eigenvectors will then be used to calculate the sensitivity and elasticity values for the matrix elements.

Objectives: This exercise will familiarize you with matrix multiplication and the basic properties of projection matrices. The exercise will also illustrate the value of working through a published example to learn a new method. The example that will be used in this exercise is drawn from the paper by Hiraldo et al. (1996) on Lesser Kestrels that is in your reader. This exercise will extend your use of program Excel, particularly moving values among sheets within one file.

1. Start up Excel. In the new file, double-click on the tabs that say 'Sheet1' and 'Sheet2' and rename them 'Rates' and 'Power'. Right-click on 'Sheet3' and delete it.

2. Set up the Rates sheet with the vital rates.
Enter the following headers:
A7 = Vital rates, B7 = Symbols, C7=Values
In B8 to B12, enter the symbols

b
co
ca
so
sa

which correspond to fecundity, breeding propensity of juveniles, breeding propensity of adults, survival of juveniles and survival of adults.
Put these labels in A8 to A12 if you like.
In C8 to C12, enter the values for these vital rates (Table 6 of Hiraldo et al.).

0.9321
0.3847
0.9250
0.3409
0.7101

3. Set up the matrix (equation 2 of Hiraldo et al.).

$$\mathbf{A} = \begin{bmatrix} co \times b \times so & ca \times b \times so \\ sa & sa \end{bmatrix}$$

Put the label 'Matrix' in A16, and do the calculations to create the matrix in the four cells spanning B16 to C17.

4. Now we will raise the matrix to higher powers using the rules of matrix multiplication. Recall that if:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

then

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} (a \times e) + (b \times g) & (a \times f) + (b \times h) \\ (c \times e) + (d \times g) & (c \times f) + (d \times h) \end{bmatrix}$$

Go to the sheet named 'Power' and enter the labels A6 = Original matrix, C6 = Exponent
Now pull over the values from the rates sheet by entering:
=Rates!B16 into A7
=Rates!C16 into B7
=Rates!B17 into A8
=Rates!C17 into B8
=1 into C7

Next you will multiply the matrix by itself by using the multiplication rules above. To get started, enter the following:
= (A7*A7 + B7*A8) into A10
and =C7+1 into C10
Complete the calculations for the rest of the cells of this matrix.

When you are finished, copy the set of lines from A10 to C12 down until you have raised the matrix to the 50^{th} power. This final matrix should occupy cells A154 to C155. This is called the power method and works because the proportions (but not the absolute values) eventually stabilize in the cells of the matrix.

5. Now we are ready to calculate the dominant eigenvalue and eigenvectors of the matrix. Enter the following labels A157 = Lambda, A158 = RV, A160 = SAD, A162 = Sensitivity, A164 = Elasticity

The rate of population change can be estimated as the ratio of any two elements in the final and penultimate matrices. Enter =A154/A151 into C157. See if you get a different value for any other pair of elements.

The columns of the final matrix give the reproductive values (RV). Remember that RV are usually scaled with respect to the RV of yearlings. Enter into C158: = (A154+A155)/ (A154+A155). What should be the calculation for cell C159?

The rows of the final matrix give the stable age distribution (SAD). Remember that the SAD should sum to one. Enter into C160: (A154 + B154) / sum(A154:B155)
What should be the calculation for cell C161?

6. The final step is to combine the eigenvectors to calculate the sensitivity and elasticity matrices.

The sensitivity matrix: $$s_{ij} = \frac{\partial \lambda}{\partial a_{ij}} = \frac{v_i w_i}{\langle \mathbf{w}, \mathbf{v} \rangle}$$

The denominator $\langle \mathbf{w}, \mathbf{v} \rangle$ is a scalar product and is equal to =((C158*C160)+(C159*C161))
The order of the terms in the numerator is quite specific and $\neq w_i v_i$.
The numerator in cell B162 = C158*C160 divided by the denominator above and C162 = C158*C161 divided by the denominator above. What are the formulae for the other two cells B163 and C163?

The elasticity matrix (e_{ij}) is easier to calculate: $$e_{ij} = \frac{\partial \ln \lambda}{\partial \ln a_{ij}} = \frac{a_{ij}}{\lambda} \frac{v_i w_i}{\langle \mathbf{w}, \mathbf{v} \rangle}$$

Remember that a_{ij} indicates an element of row *i* and column *j* in the matrix. Can you figure out the calculations necessary to create the elasticity matrix and put it in cells B164 to C165?

7. Summary steps.

Go back to the rates sheet and cell A18. In this cell, enter =PowerA157. Copy this cell to fill the block from A18 to C26. Now you have all the input rates and summary statistics on one page to view at the same time.

8. Double-check your work against the following values:

Matrix		0.1222	0.2939
		0.7101	0.7101
Lambda			0.9594
RVw	**Y**		1.0000
	A		1.1789
SAD	**Y**		0.2599
	A		0.7401
S		0.2295	0.6536
		0.2705	0.7705
E		0.0292	0.2002
		0.2002	0.5703